Module categories and Auslander-Reiten theory for generalized Beilinson algebras

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Zusammenfassung

Die Darstellungstheorie beschäftigt sich mit dem Studium von Moduln über einer gegebenen Algebra. Zwei Klassen von Algebren, die oft betrachtet werden, sind Gruppenalgebren und Algebren von endlicher globaler Dimension, insbesondere erbliche Algebren. In beiden Fällen ist es oft hoffnungslos, alle unzerlegbaren Darstellungen bestimmen zu wollen, das heißt, die Algebra ist wild. Carlson, Friedlander und Pevtsova haben 2008 die Klasse der Moduln von konstantem Jordan-Typ über einer gegebenen Gruppenalgebra eingeführt. Eine echte Unterklasse davon bilden die Moduln mit gleichen Bildern. Es stellt sich jedoch heraus, dass auch diese beiden Modulklassen im Allgemeinen sehr kompliziert sind.
Gruppenalgebren elementar abelscher p-Gruppen sind von besonderem Interesse in der modularen Darstellungstheorie endlicher Gruppen. Sie stellen die minimalen Beispiele wilder Gruppenalgebren dar, und darüberhinaus lässt sich nach einem Satz von Chouinard die Projektivität von Moduln über einer gegebenen Gruppenalgebra durch Restriktionen auf elementar abelsche p-Gruppen testen. Carlson, Friedlander und Suslin haben Moduln mit gleichen Bildern über $k(\mathbb{Z}_p \times \mathbb{Z}_p)$ studiert und die sogenannten W-Moduln, eine Beispielklasse von Moduln mit gleichen Bildern, definiert. Die unzerlegbaren $k(\mathbb{Z}_p \times \mathbb{Z}_p)$-Moduln von Loewy-Länge zwei lassen sich mit den unzerlegbaren Moduln über der erblichen Wegealgebra des Kronecker-Köchers identifizieren. Hierbei entsprechen die Moduln mit gleichen Bildern den präinjektiven Moduln.

Diese Arbeit ist dadurch motiviert, Moduln mit gleichen Bildern über $k(\mathbb{Z}_p^{\times r})$ für beliebiges r verstehen zu wollen. Wir geben eine Verallgemeinerung der W-Moduln an. Um $k(\mathbb{Z}_p^{\times r})$-Moduln mit beschränkter Loewy-Länge zu studieren, betrachten wir verallgemeinerte Beilinson-Algebren. Die Existenz eines treuen, exakten Funktors von der Modulkategorie der verallgemeinerten Beilinson-Algebra $B(n, r)$ auf n Knoten in die Modulkategorie von $k(\mathbb{Z}_p^{\times r})$ lässt uns Moduln mit gleichen Bildern und Moduln von konstantem Jordan-Typ über $B(n, r)$ definieren. Ein Hauptresultat ist die homologische Charakterisierung dieser Modulkategorien über eine Familie von Moduln mit projektiver Dimension eins.
Der sogenannte Auslander-Reiten-Köcher ist ein Hilfsmittel, um die Modulkategorie einer Algebra zu strukturieren. Seine Knoten entsprechen den Isomorphieklassen unzerlegbarer Moduln und seine Pfeile irreduziblen Abbildungen. Wir zeigen, dass die Klasse der Moduln mit gleichen Bildern über $B(n, r)$ eine Torsionsklasse mit speziellen Eigenschaften ist. Dadurch können wir Beobachtungen bezüglich der relativen Position von Moduln mit gleichen Bildern im Auslander-Reiten-Köcher $\Gamma(n, r)$ von $B(n, r)$ anstellen. Bislang ist über die Zusammenhangskomponenten von $\Gamma(n, r)$ wenig bekannt. Wir zeigen dass die verallgemeinerten W-Moduln in $\mathbb{Z}A_\infty$-Komponenten liegen, in denen alle Moduln von konstantem Jordan-Typ sind.
Da die Algebra $B(2, r)$ isomorph zur verallgemeinerten Kronecker-Algebra ist, können wir Methoden aus der Theorie der wilden erblichen Algebren anwenden und zeigen, dass für $r > 2$ jede reguläre Komponente von $\Gamma(2, r)$, im Gegensatz zum Fall $r = 2$, unendlich viele Moduln mit gleichen Bildern enthält.

Abstract

Representation theory is concerned with understanding the modules over a given algebra. Two classes of algebras that are frequently studied are group algebras and algebras of finite global dimension, in particular hereditary algebras. In both settings, it is often the case that it is not possible to classify all indecomposable representations, i.e. the algebra is wild. In 2008, Carlson, Friedlander and Pevtsova introduced the class of modules of constant Jordan type as a subclass of the module category over a given group algebra. There is the more restrictive notion of modules with the equal images property. It turns out, however, that these module classes are still very complicated in general.
Group algebras of elementary abelian p-groups are of immediate interest in the modular representation theory of finite groups. In a way, they constitute the smallest examples of wild group algebras and by Chouinard's theorem, projectivity of modules over a given group algebra can be tested via restrictions to elementary abelian p-groups. Carlson, Friedlander and Suslin have studied modules with the equal images property and modules of constant Jordan type over $k(\mathbb{Z}_p \times \mathbb{Z}_p)$. They introduce the so-called W-modules which are prominent examples of modules with the equal images property. The indecomposable $k(\mathbb{Z}_p \times \mathbb{Z}_p)$-modules of Loewy length two can be identified with the indecomposables over the hereditary Kronecker algebra, where modules with the equal images property correspond to the preinjective modules.

This thesis is inspired by the aim to understand modules with the equal images property over $k(\mathbb{Z}_p^{\times r})$ for arbitrary r. We give a generalization of the W-modules. In order to study $k(\mathbb{Z}_p^{\times r})$-modules with restricted Loewy length, we introduce generalized Beilinson algebras. Making use of a faithful exact functor from the module category of the generalized Beilinson algebra $B(n,r)$ on n vertices into the module category of the group algebra, we define the constant Jordan type property and the equal images property for modules over $B(n,r)$. A main achievement is that we are able to give a homological characterization of these subcategories by a family of modules of projective dimension one.
A tool to organize a given module category is provided by the Auslander-Reiten quiver whose vertices correspond to isomorphism classes of indecomposable objects and whose arrows correspond to irreducible maps between indecomposables. By showing that the class of modules with the equal images property over $B(n,r)$ is a torsion class with special properties, we are able to make statements about the relative position of modules with the equal images property in the Auslander-Reiten quiver $\Gamma(n,r)$ of $B(n,r)$. So far, not much is known about the connected components of $\Gamma(n,r)$. We show that generalized W-modules determine $\mathbb{Z}A_\infty$-components of $\Gamma(n,r)$ that entirely consist of modules with the constant Jordan type property.
Due to the fact that the algebra $B(2,r)$ is isomorphic to the generalized Kronecker algebra, we can apply methods from the theory of wild hereditary algebras and show that each regular $\mathbb{Z}A_\infty$-component of $\Gamma(2,r)$, $r > 2$, contains infinitely many modules with the equal images property, contrasting the findings for $r = 2$.

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Contents

1 Introduction and notation

1.1 Introduction

Addressing the study of representations of a finite group scheme G over an algebraically closed field k of characteristic $p > 0$, Friedlander and Pevtsova have established the notion of so-called p-points in [17]. These are certain embeddings $\alpha\colon k[T]/(T^p) \to kG$ along which representations of the group algebra kG can be restricted to the less complicated algebra $k\mathbb{Z}_p \cong k[T]/(T^p)$. The representations of $k[T]/(T^p)$ are completely understood in terms of Jordan block decompositions, whereas the algebra kG is wild in most cases, i.e. there is no hope to classify all indecomposable modules. Hence it is reasonable to study representations with additional properties. In [10], Carlson, Friedlander and Pevtsova have introduced the class of modules of constant Jordan type. A finite-dimensional kG-module M has constant Jordan type if the Jordan block decomposition of the pullback $\alpha^*(M)$ does not depend on the choice of the p-point α [10, 1.10]. There is the refined notion of the constant j-rank property such that a module has constant Jordan type iff it has constant j-rank for all $j \geq 1$ (cf. [18, p. 11]).

Confining investigations to elementary abelian p-groups $E_r = (\mathbb{Z}_p)^{\times r}$ of rank $r \geq 2$, a more restrictive condition has been formulated in [11] by Carlson, Friedlander and Suslin, where $M \in \operatorname{mod} kE_r$ satisfies the so-called equal images property if there exists a k-space V such that $\alpha(t).M = V$ for all p-points α with $t := T + (T^p) \in k[T]/(T^p)$. The dual concept is referred to as the equal kernels property. In [11], the authors are mainly concerned with the case $r = 2$ and they introduce a family of kE_2-modules, the so-called W-modules, which satisfy the equal images property and are ubiquitous in the sense that every module satisfying the equal images property is a quotient of a W-module [11, 4.4]. This relies on the fact that the indecomposable equal images modules of Loewy length two over kE_2 are W-modules [11, 4.1]. Moreover, when identifying kE_2-modules of Loewy length two with modules over the Kronecker algebra, the equal images modules correspond to the preinjective modules (cf. [16, 4.2.2]). We give a generalization of the W-modules to elementary abelian p-groups of arbitrary rank.

The approach in this thesis is motivated by the objective to understand $\operatorname{mod}_n kE_r$, i.e. the full subcategory of kE_r-modules with Loewy length bounded by $n \leq p$. With this in mind, we consider the generalized Beilinson algebra $B(n, r)$ defined via the quiver

$$\begin{tikzcd} 1 \arrow[r, bend left, "\gamma_1"] \arrow[r, bend right, "\gamma_r"'] \arrow[r, phantom, "\vdots"] & 2 \arrow[r, bend left, "\gamma_1"] \arrow[r, bend right, "\gamma_r"'] \arrow[r, phantom, "\vdots"] & 3 & \cdots & n-1 \arrow[r, bend left, "\gamma_1"] \arrow[r, bend right, "\gamma_r"'] \arrow[r, phantom, "\vdots"] & n \end{tikzcd}$$

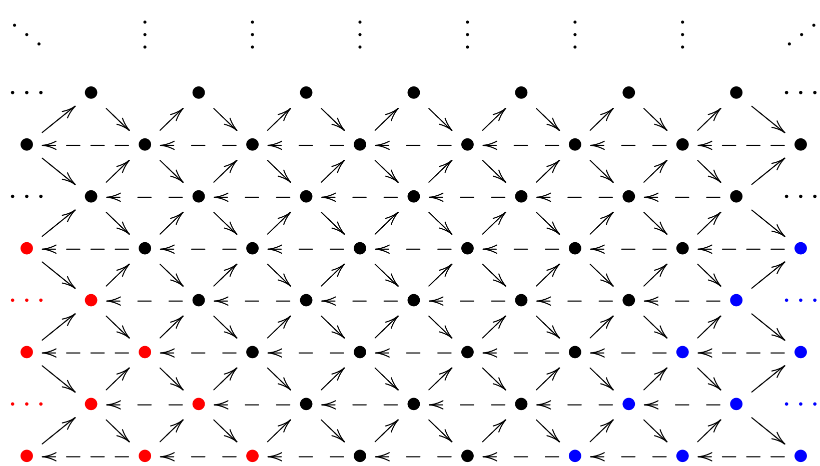
modulo commutativity relations $\gamma_i\gamma_j = \gamma_j\gamma_i$. This algebra is a two-parameter version of the Beilinson algebra $B(n) = B(n, n)$ which was first considered by Beilinson in his investigations of coherent sheaves on projective space [4].

Exploiting a faithful exact functor $\mathfrak{F}\colon \operatorname{mod} B(n,r) \to \operatorname{mod}_n(kE_r)$, we formulate analogs of the constant Jordan type and constant j-rank property as well as the equal images and equal kernels property for $B(n,r)$-modules and we define full subcategories $\mathrm{CJT}(n,r)$, $\mathrm{CR}^j(n,r)$, $\mathrm{EIP}(n,r)$, $\mathrm{EKP}(n,r) \subset \operatorname{mod} B(n,r)$ such that the restrictions of $\mathfrak{F}$ to $\mathrm{EIP}(n,r)$ and $\mathrm{EKP}(n,r)$ reflect isomorphisms and have an essential image consisting of standardly gradable modules with the equal images property and costandardly gradable modules with the equal kernels property, respectively. In particular, we may consider the generalized W-modules as modules in $\mathrm{EIP}(n,r)$.

An immediate advantage of passing over to $B(n,r)$ is that we are able to give a homological characterization of the categories $\mathrm{EIP}(n,r)$ and $\mathrm{EKP}(n,r)$ involving a $\mathbb{P}^{r-1}$-family of $B(n,r)$-modules of projective dimension one. This allows us to apply general methods from Auslander-Reiten theory and with our homological tool in hand, we prove:

Theorem (A). *The category* $\mathrm{EIP}(n,r)$ *is image- and extension-closed, closed under direct sums and thus constitutes the torsion class* $\mathcal{T}$ *of a torsion pair* $(\mathcal{T},\mathcal{F})$ *in* $\operatorname{mod} B(n,r)$. *Furthermore, we have* $\mathrm{EKP}(n,r) \subset \mathcal{F}$ *and* $\mathcal{T}$ *is closed under the Auslander-Reiten translate* τ *and contains all preinjective modules.*
Dually, $\mathrm{EKP}(n,r)$ *is the torsion-free class* $\mathcal{F}'$ *of a torsion pair* $(\mathcal{T}',\mathcal{F}')$ *in* $\operatorname{mod} B(n,r)$ *such that* $\mathrm{EIP}(n,r) \subset \mathcal{T}'$ *and* $\mathcal{F}'$ *is closed under* τ^{-1} *and contains all preprojective modules.*
In particular, there are no non-trivial maps $\mathrm{EIP}(n,r) \to \mathrm{EKP}(n,r)$.

We want to determine the relative position of these modules in the Auslander-Reiten quiver $\Gamma(n,r)$ of $B(n,r)$. The algebra $B(2,r)$ is isomorphic to the generalized Kronecker algebra and hence all regular components of $\Gamma(2,r)$, $r > 2$, are of type $\mathbb{Z}A_\infty$ as proven by Ringel in [35]. For $n > 2$, however, not much is known about the shape of the connected components of $\Gamma(n,r)$. An immediate consequence of Theorem (A) is that in case $\mathcal{C}$ is a $\mathbb{Z}A_\infty$-component of $\Gamma(n,r)$ such that $\mathrm{EIP}(n,r) \cap \mathcal{C}$ and $\mathrm{EKP}(2,r) \cap \mathcal{C}$ are non-empty, these sets form disjoint cones:



The red and blue bullets indicate that the corresponding module is an object in $\mathrm{EIP}(n,r)$, respectively in $\mathrm{EKP}(n,r)$. The size of the gap $\mathcal{W}(\mathcal{C})$ between these two cones is an invariant of $\mathcal{C}$. We prove that if $\mathcal{W}(\mathcal{C}) = 0$, then all modules in $\mathcal{C}$ satisfy the constant Jordan type property. Applying results by Kerner [26] on wild hereditary algebras, we are able to make explicit statements about the occurrence of modules with the equal images property in $\Gamma(2,r)$, $r > 2$, and contrast the findings for $r = 2$.
Interpreting $B(n,r)$, $n \geq 3$, as an iterated one-point extension of the r-Kronecker algebra by duals of generalized W-modules, we obtain further information concerning the occurrence of the corresponding modules in $\Gamma(n,r)$.

Theorem (B). *Let $r \geq 2$, $m > n \geq 2$.*

(a) If $r > 2$, then

- *(i) each regular $\mathbb{Z}A_\infty$-component $\mathcal{C}$ of $\Gamma(2,r)$ contains non-empty disjoint cones $\mathrm{EIP}(2,r) \cap \mathcal{C}$ and $\mathrm{EKP}(2,r) \cap \mathcal{C}$.*
- *(ii) for each $d \in \mathbb{N}$, there exists a regular component $\mathcal{C}$ of $\Gamma(2,r)$ such that $\mathcal{W}(\mathcal{C}) > d$.*

(b) If $r > 2$ or $n > 2$, then the generalized W-module $W_{m,n}^{(r)}$ belongs to a $\mathbb{Z}A_\infty$-component $\mathcal{C}_m^{(r)}$ of $\Gamma(n,r)$ with $\mathcal{C}_m^{(r)} \subseteq \mathrm{CJT}(n,r)$.

We thus show that there is a great supply of indecomposable modules with the equal images and the constant Jordan type property over kE_r and that the Auslander-Reiten quiver $\Gamma(n,r)$ proves to be a suitable tool to organize these categories. Furthermore, studying the torsion class $\mathrm{EIP}(n,r)$ and its corresponding torsion-free class enables us to make general statements concerning the Auslander-Reiten theory of $B(n,r)$.

This thesis is organized as follows: In Chapter 2, we recall definitions and basic results and give a generalization of the W-modules defined in [11] to arbitrary rank. We introduce generalized Beilinson algebras and give a homological description of the categories $\mathrm{CJT}(n,r)$, $\mathrm{CR}^j(n,r)$, $\mathrm{EIP}(n,r)$ and $\mathrm{EKP}(n,r)$ in Chapter 3 and point out the special role that generalized W-modules play in $\mathrm{EIP}(n,r)$. Moreover, we study restrictions of $B(n,r)$-modules to $B(k,r)$, $k < n$, whereby we can show that certain torsion-free modules determine $\mathbb{Z}A_\infty$-components in $\Gamma(n,r)$. In Chapter 4, we restrict our investigations to modules of Loewy length two and give our more specific results on the r-Kronecker together with some examples. In Chapter 5, we make use of the theory of one-point extensions to determine the occurrence of generalized W-modules in $\Gamma(n,r)$.

1.2 Notation and Prerequisites

For convenience, we will briefly recall some of the main concepts and results we will use in this thesis. Moreover, we will introduce our notation. A thorough introduction to the representation theory of associative algebras may be found in [2], [3] or [5], for example.

Throughout, k denotes an algebraically closed field. When studying elementary abelian p-groups, we require that $\operatorname{char}(k) = p > 0$. We may drop this assumption, however, when dealing with modules over the polynomial ring as well as over generalized Beilinson algebras.

A k-algebra A is always assumed to be associative and unitary. When speaking of an A-module, we mean a left A-module. The category of all finitely generated A-modules will be denoted by $\operatorname{mod} A$. Given $V \in \operatorname{mod} k$, we denote by V^* its k-linear dual $\operatorname{Hom}_k(V, k) \in \operatorname{mod} k$.

1.3 Graded algebras

We will make use of the graded structure of certain algebras. $\mathbb{Z}$-graded (Artin) algebras and their modules categories were thoroughly studied by Gordon and Green in [20] and [21]. Generalizations to $\mathbb{Z}^n$-graded algebras can be found in [15]. An introduction to the theory of Koszul algebras is given by Martínez-Villa in [30]. Proofs of the facts stated in this subsection can be found in [20], [21], [15] and [30], respectively.

Definition 1.1. *A k-algebra A is* **$\mathbb{Z}^n$-graded** *for some $n \in \mathbb{N}$ if A affords a vector space decomposition $A = \bigoplus_{i \in \mathbb{Z}^n} A_i$ such that $A_i A_j \subseteq A_{i+j}$ for all $i, j \in \mathbb{Z}^n$.*

Definition 1.2. *Let $A = \bigoplus_{i \in \mathbb{Z}^n} A_i$ be a $\mathbb{Z}^n$-graded k-algebra.*

(i) An ideal $I \subseteq A$ is called **homogeneous** *if $I = \bigoplus_{i \in \mathbb{Z}^n} I_i$ with $I_i \subseteq A_i$.*

(ii) A module $M \in \operatorname{mod} A$ is called **$\mathbb{Z}^n$-graded** *if there exists a vector space decomposition $M = \bigoplus_{j \in \mathbb{Z}^n} M_j$ such that $A_i M_j \subseteq M_{i+j}$ for all $i, j \in \mathbb{Z}^n$.*

(iii) The category $\operatorname{mod}_{\mathbb{Z}^n} A$ has the finitely generated $\mathbb{Z}^n$-graded A-modules as objects and for $M = \bigoplus_{j \in \mathbb{Z}^n} M_j,\ N = \bigoplus_{j \in \mathbb{Z}^n} N_j \in \operatorname{mod}_{\mathbb{Z}^n} A$, the set of morphisms $\operatorname{Hom}_A^{\mathbb{Z}^n}(M, N)$ consists of the A-linear maps $\varphi : M \to N$ with $\varphi(M_j) \subseteq N_j$ for all $j \in \mathbb{Z}^n$.

(iv) Let $M = \bigoplus_{j \in \mathbb{Z}^n} M_j \in \operatorname{mod}_{\mathbb{Z}^n} A$. Then $\operatorname{supp}(M) = \{j \in \mathbb{Z}^n \mid M_j \neq 0\}$ is called the **support** *of M.*

(v) Given $i \in \mathbb{Z}^n$, the **i-th shift functor** *$[i]\colon \operatorname{mod}_{\mathbb{Z}^n} A \to \operatorname{mod}_{\mathbb{Z}^n} A$ associates to each $M \in \operatorname{mod}_{\mathbb{Z}^n} A$ the object $M[i] \in \operatorname{mod}_{\mathbb{Z}^n} A$ with $M[i]_j := M_{j-i}$ for all $j \in \mathbb{Z}^n$ while morphisms are left unchanged.*

(vi) Let $n = 1$ and $M = \bigoplus_{j \in \mathbb{Z}} M_j \in \operatorname{mod}_{\mathbb{Z}} A$ such that $M_j = 0$ whenever $j < 0$. For $i \in \mathbb{Z}$, we denote by $M_{\geq i}$ the submodule $\bigoplus_{j \geq i} M_j \subseteq M$ and by $M_{<i}$ the factor module $M/M_{\geq i}$ in $\operatorname{mod}_{\mathbb{Z}} A$. Given furthermore $N = \bigoplus_{j \in \mathbb{Z}} N_j \in \operatorname{mod}_{\mathbb{Z}} A$ with $N_j = 0$ whenever $j < 0$, for $\varphi \in \operatorname{Hom}_A^{\mathbb{Z}}(M, N)$, we define
$\varphi_{\geq i}\colon M_{\geq i} \to N_{\geq i},\ m \mapsto \varphi(m)$ and $\varphi_{<i} : M_{<i} \to N_{<i},\ m + M_{\geq i} \mapsto \varphi(m) + N_{\geq i}$.

The faithful exact **forgetful functor**

$$\mathfrak{F} : \operatorname{mod}_{\mathbb{Z}^n} A \to \operatorname{mod} A$$

simply forgets the grading on objects.

Definition 1.3. *Let $A = \bigoplus_{i\in\mathbb{Z}^n} A_i$ be a $\mathbb{Z}^n$-graded k-algebra, $M \in \operatorname{mod} A$ and $J \subseteq \mathbb{Z}^n$.*

(i) We call M **gradable** *if $M \cong \mathfrak{F}(\bigoplus_{j\in\mathbb{Z}^n} M_j)$ for some $\bigoplus_{j\in\mathbb{Z}^n} M_j \in \operatorname{mod}_{\mathbb{Z}^n} A$.*

(ii) We call M **J-gradable** *if $M \cong \mathfrak{F}(\bigoplus_{j\in\mathbb{Z}^n} M_j)$ for some $\bigoplus_{j\in\mathbb{Z}^n} M_j \in \operatorname{mod}_{\mathbb{Z}^n} A$ with $\operatorname{supp}(\bigoplus_{j\in\mathbb{Z}^n} M_j) \subseteq J$.*

The functor $\mathfrak{F}$ has nice properties:

Proposition 1.4. *Let $M, N \in \operatorname{mod}_{\mathbb{Z}^n} A$.*

(i) The module $\mathfrak{F}(M)$ is indecomposable if and only if M is indecomposable.

(ii) If M is indecomposable, then

$$\mathfrak{F}(M) \cong \mathfrak{F}(N)$$

if and only if $N = M[i]$ for some $i \in \mathbb{Z}^n$.

Note that if $M, N \in \operatorname{mod}_{\mathbb{Z}^n} A$ afford gradings $M = \bigoplus_{j\in\mathbb{Z}^n} M_j$ and $N = \bigoplus_{j\in\mathbb{Z}^n} N_j$, then by setting

$$\operatorname{Hom}_A(\mathfrak{F}(M), \mathfrak{F}(N))_i = \{\varphi \in \operatorname{Hom}_A(\mathfrak{F}(M), \mathfrak{F}(N)) \mid \forall j \in \mathbb{Z}^n : \varphi(M_j) \subseteq N_{i+j}\}$$

for all $i \in \mathbb{Z}^n$, the space $\operatorname{Hom}_A(\mathfrak{F}(M), \mathfrak{F}(N))$ is endowed with a grading

$$\operatorname{Hom}_A(\mathfrak{F}(M), \mathfrak{F}(N)) = \bigoplus_{i\in\mathbb{Z}^n} \operatorname{Hom}_A(\mathfrak{F}(M), \mathfrak{F}(N))_i$$

as a module over the $\mathbb{Z}^n$-graded algebra $\operatorname{End}_A(\mathfrak{F}(N))$.

Definition 1.5. *We call a graded algebra $A = \bigoplus_{i\in\mathbb{Z}} A_i$* **standardly graded***, provided*

(i) A_i is finite-dimensional for all $i \in \mathbb{Z}$,

(ii) $A_i = 0$ for $i < 0$,

(iii) $A_0 = k \times \cdots \times k$ as k-algebras,

(iv) for all $i, j \in \mathbb{Z}$, we have $A_i A_j = A_{i+j}$.

We denote by $J(A) = \bigoplus_{i>0} A_i$ the **graded radical** *of A.*

Definition 1.6. *Let A be a standardly graded k-algebra. We call $M = \bigoplus_{j\in\mathbb{Z}} M_j \in \operatorname{mod}_{\mathbb{Z}} A$*

(i) **standardly graded** *if M is generated by M_i, where $i = \min\operatorname{supp}(M)$.*

(ii) **costandardly graded** *if M is cogenerated by M_i, where $i = \max\operatorname{supp}(M)$.*

We refer to the essential images of such modules under $\mathfrak{F} : \operatorname{mod}_{\mathbb{Z}} A \to \operatorname{mod} A$ as **standardly gradable** *and* **costandardly gradable** *modules, respectively.*

Definition 1.7. *Let A be a standardly graded algebra. We call $M \in \operatorname{mod}_{\mathbb{Z}} A$* **Koszul**, *if M has a graded projective resolution*

$$\cdots P^n \xrightarrow{\delta_n} \cdots \xrightarrow{\delta_2} P^1 \xrightarrow{\delta_1} P^0 \xrightarrow{\delta_0} M \to 0$$

such that for each j, the projective module $P^j \in \operatorname{mod}_{\mathbb{Z}} A$ is generated by $(P^j)_j$ and the δ_i are maps of degree 0.
The algebra A is called **Koszul** *provided all graded simple A-modules generated in degree 0 are Koszul.*

Proposition 1.8. *Let A be a Koszul algebra and let $M \in \operatorname{mod}_{\mathbb{Z}} A$ be Koszul. Then $J(A)M[-1]$ is also Koszul.*

1.4 Algebras given by quivers and relations

We assume that the reader is familiar with the notion of algebras that are defined via quivers with relations and simply introduce our notation (cf. [2, II, III]).

Definition 1.9. *Let A be a finite-dimensional algebra given by a finite quiver $Q = (Q_0, Q_1)$ with relations and $M \in \operatorname{mod} A$.*

(i) We denote by $P(i)$, $I(i)$ and $S(i)$ the projective, injective and simple module corresponding to the vertex $i \in Q_0$.

(ii) We denote by e_i the primitive idempotent corresponding to the vertex $i \in Q_0$.

(iii) We denote by $M_i = e_i.M$ the vector space corresponding to the vertex $i \in Q_0$.

(iv) We denote by $\underline{\dim} M = (\dim_k M_i)_{i \in Q_0}$ the **dimension vector** *of M.*

Given a finite quiver Q, the path algebra kQ is endowed with a standard $\mathbb{Z}$-grading

$$kQ = \bigoplus_{i \geq 0} kQ_i$$

by defining kQ_i to be the k-linear span of all paths of length $i \geq 0$. This grading is referred to as the **path-length-grading**.
If furthermore $I \subseteq kQ$ is homogeneous with respect to this grading, then kQ/I inherits the path-length-grading from kQ and is thus a standardly graded algebra.

1.5 Torsion theory

We recall the concept of a torsion pair and state basic facts on torsion theory which can be found in [2, VI.1].

Definition 1.10. *Given an algebra A, a pair $(\mathcal{T}, \mathcal{F})$ of full subcategories of $\operatorname{mod} A$ is called* **torsion pair** *if the following conditions are satisfied:*

(a) $\operatorname{Hom}_A(M,N) = 0$ *for all* $M \in \mathcal{T}$, $N \in \mathcal{F}$.

(b) $\operatorname{Hom}_A(M,-)|_{\mathcal{F}} = 0$ *implies* $M \in \mathcal{T}$.

(c) $\operatorname{Hom}_A(-,N)|_{\mathcal{T}} = 0$ *implies* $N \in \mathcal{F}$.

The categories $\mathcal{T}$ *and* $\mathcal{F}$ *are then referred to as the* **torsion class**, *respectively* **torsion-free class**, *of the torsion pair* $(\mathcal{T},\mathcal{F})$.

Torsion classes are those full subcategories of $\operatorname{mod} A$ that are closed under images, direct sums and extensions whereas torsion-free classes correspond to the full subcategories of $\operatorname{mod} A$ that are closed under submodules, direct products and extensions.

Definition 1.11. *Let* $(\mathcal{T},\mathcal{F})$ *be a torsion pair in* $\operatorname{mod} A$. *The functor* $\mathfrak{t} : \operatorname{mod} A \to \operatorname{mod} A$ *that associates to each* $M \in \operatorname{mod} A$ *the largest submodule* $\mathfrak{t}(M) \subseteq M$ *that lies in* $\mathcal{T}$ *is called the* **torsion radical**.

Proposition 1.12. *Let* $(\mathcal{T},\mathcal{F})$ *be a torsion pair in* $\operatorname{mod} A$ *and* $M \in \operatorname{mod} A$. *There exists a short exact sequence*

$$0 \to \mathfrak{t}(M) \to M \to M/\mathfrak{t}(M) \to 0$$

with $\mathfrak{t}(M) \in \mathcal{T}$ *and* $M/\mathfrak{t}(M) \in \mathcal{F}$.
This sequence is unique in the sense that if $0 \to M' \to M \to M'' \to 0$ *is exact with* $M' \in \mathcal{T}$ *and* $M'' \in \mathcal{F}$, *then the two sequences are isomorphic.*

Definition 1.13. *A torsion pair is called* **split** *if the above sequence splits for all* $M \in \operatorname{mod} A$.

1.6 Auslander-Reiten theory

Given a finite-dimensional algebra A, the module category $\operatorname{mod} A$ can be described in terms of the Auslander-Reiten quiver $\Gamma(A)$ which is defined via so-called irreducible maps. For more detailed information and proofs, the reader is referred to [2, IV, A.3.].

Definition 1.14. *Let* $M, N \in \operatorname{mod} A$. *A morphism* $f\colon N \to M$ *is called* **irreducible** *if* f *is neither a split monomorphism nor a split epimorphism and whenever* $f = f_1 f_2$, *then* f_1 *is a split epimorphism or* f_2 *is a split monomorphism.*

Lemma 1.15. *Let* $M, N \in \operatorname{mod} A$ *be indecomposable. Then* $f\colon M \to N$ *is irreducible if and only if* $f \in \operatorname{rad}_A(M,N) \setminus \operatorname{rad}_A^2(M,N)$.

Here, rad_A denotes the radical of the category $\operatorname{mod} A$. In view of Lemma 1.15, we define the following:

Definition 1.16. *Let* $M, N \in \operatorname{mod} A$ *be indecomposable. Then*

$$\operatorname{Irr}(M,N) := \operatorname{rad}_A(M,N)/\operatorname{rad}_A^2(M,N)$$

is referred to as the **space of irreducible morphisms.**

Remark 1.17. An irreducible morphism $f : N \to M$ is either a proper monomorphism or a proper epimorphism.

The **Auslander-Reiten quiver** is given by the following data:

(i) The vertices $\Gamma(A)_0$ correspond to the isomorphism classes $[M]$ of indecomposable A-modules.

(ii) The arrows from $[N]$ to $[M]$ in $\Gamma(A)_1$ correspond to a basis of $\mathrm{Irr}(N, M)$.

Each non-projective indecomposable module M (non-injective indecomposable module N) gives rise to a uniquely determined short exact sequence, an **Auslander-Reiten** (or **almost split**) **sequence**,

$$0 \to N \xrightarrow{f} \bigoplus_{i=1}^{t} E_i^{n_i} \xrightarrow{g} M \to 0$$

where N (M) is indecomposable, the E_i are pairwise non-isomorphic and indecomposable and the maps $f_{i,1}, \dots, f_{i,n_i} \colon N \to E_i$, $g_{i,1}, \dots, g_{i,n_i} \colon E_i \to M$ are bases of the vector spaces $\mathrm{Irr}(N, E_i)$ and $\mathrm{Irr}(E_i, M)$, respectively.
The module N is referred to as the **Auslander-Reiten translation** of M and we denote this in $\Gamma(A)$ by $[N] \dashleftarrow [M]$. The Auslander-Reiten translation can be computed as follows: for $M \in \operatorname{mod} A$ (not necessarily indecomposable), choose a minimal projective presentation

$$P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \to 0$$

and apply the functor $\mathrm{Hom}_A(-, A)$ to this sequence. Then $\mathrm{Tr}(M) := \operatorname{coker} \mathrm{Hom}_A(p_1, A)$ is called the **trace** of M and in case M is indecomposable, $\mathrm{Hom}_k(\mathrm{Tr}(M), k)$ is isomorphic to the starting term of the almost split sequence ending in M. We define $\tau M = \mathrm{Hom}_k(\mathrm{Tr}(M), k)$ and $\tau^{-1}M = \mathrm{Tr}(\mathrm{Hom}_k(M, k))$.

An important tool, that we will use throughout, is the so-called Auslander-Reiten formula.

Theorem 1.18. *Let A be a k-algebra and $M, N \in \operatorname{mod} A$. Then there exist isomorphisms*

$$\mathrm{Ext}^1_A(M, N) \cong \left(\underline{\mathrm{Hom}}_A(\tau^{-1}N, M)\right)^* \cong \left(\overline{\mathrm{Hom}}_A(N, \tau M)\right)^*.$$

Note that given $M, N \in \operatorname{mod} A$, we have $\underline{\mathrm{Hom}}_A(N, M) := \mathrm{Hom}(N, M)/\mathcal{P}(N, M)$ and $\overline{\mathrm{Hom}}_A(N, M) := \mathrm{Hom}(N, M)/\mathcal{I}(N, M)$, where $\mathcal{P}(N, M)$ and $\mathcal{I}(N, M)$ denote the morphisms $f : N \to M$ that factor through projective and injective modules, respectively.

Definition 1.19. *An indecomposable module $M \in \operatorname{mod} A$ is called*

(i) **preprojective** *if there is $n \in \mathbb{N}_0$ such that $\tau^n M$ is projective.*

(ii) **preinjective** *if there is $n \in \mathbb{N}_0$ such that $\tau^{-n} M$ is injective.*

(iii) **regular** *if M is neither preinjective nor preprojective.*

Connected components of $\Gamma(A)$ that consist entirely of preprojective, preinjective or regular modules are then called **preprojective**, **preinjective** *and* **regular components**, *respectively.*

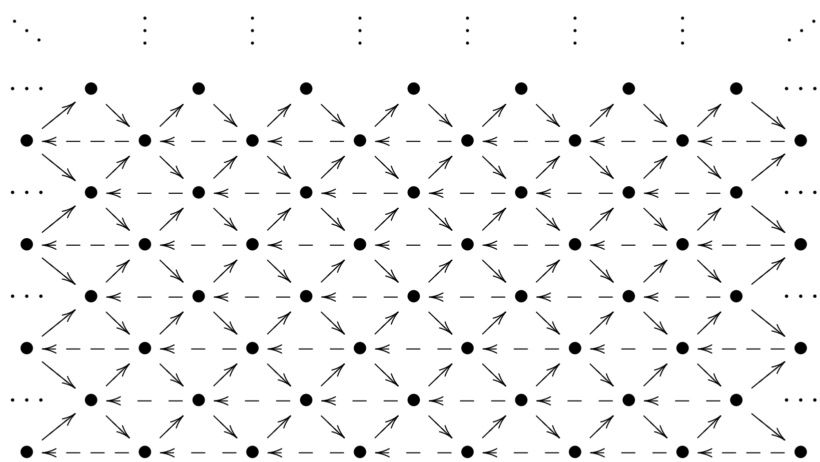
Definition 1.20. *Let $M, N \in \operatorname{mod} A$ be indecomposable. We say that*

(i) N is a **predecessor** *of M if there is a directed path from $[N]$ to $[M]$ in $\Gamma(A)$, i.e. a chain of irreducible maps from N to M.*

(ii) N is a **successor** *of M if there is a directed path from $[M]$ to $[N]$ in $\Gamma(A)$, i.e. a chain of irreducible maps from M to N.*

We denote the subset of $\Gamma(A)_0$ consisting of M and all its predecessors by $(\to M)$ and the set consisting of M and all its successors by $(M \to)$, respectively.

We are particularly interested in connected components of $\Gamma(A)$ that are of type $\mathbb{Z}A_\infty$, i.e. components of the form



Modules in the bottom row of such components are called **quasi-simple**. Ringel [35] has shown that for each module M in a regular $\mathbb{Z}A_\infty$-component $\mathcal{C}$, there exist uniquely determined quasi-simple modules X and $Y \in \mathcal{C}$ with the property that there are (uniquely determined) chains of irreducible monomorphisms $X = X_1 \to \cdots \to X_{s-1} \to X_s = M$ and epimorphisms $M = Y_s \to Y_{s-1} \to \cdots \to Y_1 = Y$. We call s the **quasi-length** of M and the modules X and Y are referred to as the **quasi-socle** and **quasi-top** of M, respectively. Moreover, M is uniquely determined by its quasi-length and quasi-socle, respectively quasi-top, whence we write $M = X(s)$ and $M = [s]Y$.

1.7 Representation type

Given a finite-dimensional algebra A, one is interested in the question how complicated the category $\operatorname{mod} A$ is. This matter has, for example, been studied by Nazarova and Roiter in [32], by Donovan and Freislich in [13], by Drozd in [14] and by Crawley-Boevey in [12].

Definition 1.21 (cf. [5], 4.4.1)**.** *Let A be a finite-dimensional k-algebra. We say that*

(i) A is of **finite representation type** *if there are only finitely many isomorphism classes of indecomposable A-modules.*

(ii) A is of **tame representation type** *if A is not of finite representation type and for any $n \in \mathbb{N}$, there is a finite set of A-$k[T]$-bimodules M_i which are free as right $k[T]$-modules, with the property that all but a finite number of indecomposable A-modules of dimension n are of the form $M_i \otimes_{k[T]} M$ for some i, and for some indecomposable $k[T]$-module M.*

(iii) A has **wild representation type** *if there is a finitely generated A-$k\langle X, Y\rangle$-bimodule M which is free as a right $k\langle X, Y\rangle$-module such that the functor $M \otimes_{k\langle X,Y\rangle} -$ from finite-dimensional $k\langle X, Y\rangle$-modules to finite-dimensional A-modules preserves indecomposability and isomorphism classes.*

Thus if an algebra is wild, its representation theory contains the representation theory of a free algebra in two variables and hence it seems hopeless to give a classification of the indecomposable representations. Drozd has proven in [14] that over an algebraically closed field, every finite-dimensional algebra is of finite, tame or wild representation type, and these types are mutually exclusive.
In general, it is hard to determine the representation type of a given algebra. In some special cases, however, one can answer the question. Classes of algebras that are of special interest for us are group algebras and hereditary algebras.

Theorem 1.22 (Higman [24], Bondarenko and Drozd [8], Ringel [33])**.** *Let G be a finite group and k an infinite field of characteristic $p > 0$. Then*

(i) kG has finite representation type if and only if G has cyclic Sylow p-subgroups.

(ii) kG has tame representation type if and only if $p = 2$ and the Sylow 2-subgroups are non-cyclic and furthermore dihedral, semidihedral or generalized quaternion.

(iii) In all other cases, kG has wild representation type.

In the case of basic hereditary algebras, i.e. path algebras of finite quivers, the underlying quiver determines the representation type as follows:

Theorem 1.23 ((*i*): Gabriel [19], (*ii*): Donovan-Freislich [13], Nazarova [31])**.** *Let $A = kQ$ be the path algebra of a finite connected acyclic quiver Q. Then*

(i) A is representation-finite iff Q is of Dynkin type, i.e. A_n, D_n, E_6, E_7 or E_8.

(ii) A is tame iff Q is of Euclidean type, i.e. $\tilde{A}_n, \tilde{D}_n, \tilde{E}_6, \tilde{E}_7$ or $\tilde{E}_8$.

In all other cases, A is wild.

2 Representations of elementary abelian p-groups

In this section, we elaborate on an approach set forth by Carlson, Friedlander and Pevtsova in [10] addressing the study of modular representations of finite group schemes via algebraic families of restrictions to $k[T]/(T^p)$, an algebra whose representations are well understood. Motivated by the work of Carlson, Friedlander and Suslin on elementary abelian p-groups of rank two [11], we study elementary abelian p-groups of arbitrary rank and give a generalization of the so-called W-modules.

2.1 Module categories arising via p-points

We first of all introduce the set up and recall the relevant concepts and some basic results from [9], [10], [11] and [18]. In doing so, we will present some definitions in a way that is suitable for our purposes.

We let $E_r = (\mathbb{Z}_p)^{\times r}$ be an elementary abelian p-group of rank $r \geq 2$ with generators $g_1, \ldots, g_r$. Let furthermore $R = k[X_1, \ldots, X_r]$ be the polynomial ring in r variables. Sending X_i to $x_i := g_i - 1$ yields an isomorphism

$$k[X_1, \ldots, X_r]/(X_1^p, \ldots, X_r^p) \cong kE_r$$

between the truncated polynomial ring and the group algebra of E_r. Consider furthermore the ideal $I = (X_1, \ldots, X_r) \subseteq R$ generated by the monomials of degree one as well as the augmentation ideal $J = \operatorname{rad}(kE_r) = (x_1, \ldots, x_r)$ of kE_r. We denote by $\operatorname{mod}_n(kE_r) \subset \operatorname{mod} kE_r$ the full subcategory consisting of modules of Loewy length at most n.

Definition 2.1. *An algebra homomorphism $\alpha\colon k[T]/(T^p) \to kE_r$ is called a* p**-point** *if the pullback $\alpha^*(kE_r)$ is a free $k[T]/(T^p)$-module.*

Note that this is equivalent to saying that $\alpha(t)$ with $t := T + (T^p)$ is an element in $\operatorname{rad}(kE_r) \setminus \operatorname{rad}^2(kE_r)$ [11, p. 3]. Given such a p-point α, for $M \in \operatorname{mod} kE_r$, we consider the linear operator

$$\alpha(t)_M\colon M \to M,\ m \mapsto \alpha(t).m\,.$$

The Jordan canonical form of $\alpha(t)_M$ entirely determines the isomorphism type of the $k[T]/(T^p)$-module $\alpha^*(M)$.

Definition 2.2. *Let α be a p-point, $M \in \operatorname{mod} kE_r$. The sequence of sizes of Jordan blocks of $\alpha(t)_M$ is referred to as the* **Jordan type** *of M* **corresponding to** *α and we write*

$$\operatorname{JType}(\alpha, M) = a_p[p] + \cdots + a_1[1],$$

indicating that there are a_i blocks of size $[i]$ for $1 \leq i \leq p$.
If $\operatorname{JType}(\alpha, M) = \operatorname{JType}(\beta, M)$ for all p-points β, we say that M is of **constant Jordan type**

$$\operatorname{JType}(M) := \operatorname{JType}(\alpha, M).$$

Definition 2.3. *Let $j \in \mathbb{N}$. We say that $M \in \operatorname{mod} kE_r$ is of* **constant j-rank** *if*

$$\operatorname{rk} \alpha(t)^j_M = \operatorname{rk} \beta(t)^j_M$$

for all p-points α, β.

Note that M is of constant Jordan type iff M is of constant j-rank for all $j \geq 1$ [18, p. 11]. We denote the full subcategories of $\operatorname{mod} kE_r$ consisting of such modules by $\mathrm{CJT}(kE_r)$ and $\mathrm{CR}^j(kE_r)$.

Definition 2.4. *A module $M \in \operatorname{mod} kE_r$ is said to satisfy the* **equal images property** *if* $\operatorname{im} \alpha(t)_M = \operatorname{im} \beta(t)_M$ *for all p-points α and β.*
Dually, $M \in \operatorname{mod} kE_r$ is said to satisfy the **equal kernels property** *if* $\ker \alpha(t)_M = \ker \beta(t)_M$ *for all p-points α and β.*

In [11, 1.2, 1.7], it is shown that it suffices to check the above properties for all p-points α with $\alpha(t) = \alpha_1 x_1 + \cdots + \alpha_r x_r$ for a non-trivial element $(\alpha_1, \ldots, \alpha_r) \in k^r \backslash 0$ and furthermore the following holds:

Remark 2.5. If $M \in \operatorname{mod} kE_r$ satisfies the equal images property, then $\operatorname{im} \alpha(t)_M = \operatorname{rad}(M)$ for all p-points α.

We denote the corresponding full subcategories of $\operatorname{mod}(kE_r)$ by $\mathrm{EIP}(kE_r)$ and $\mathrm{EKP}(kE_r)$, respectively. Note that $\mathrm{EIP}(kE_r) \cup \mathrm{EKP}(kE_r) \subseteq \mathrm{CJT}(kE_r)$ [11, 1.9] and furthermore $\mathrm{EIP}(kE_r) \cap \mathrm{EKP}(kE_r) = \operatorname{add} k$ [16, 4.4.3], where k is the trivial kE_r-module and $\operatorname{add} k$ the full subcategory of $\operatorname{mod} kE_r$ whose objects are direct sums of the trivial module k.

Given $M \in \operatorname{mod} kE_r$, the linear dual M^* naturally carries the structure of a right kE_r-module with action given by $(f.x)(m) = f(x.m)$ for all $x \in kE_r$, $f \in \operatorname{Hom}_k(M, k)$, $m \in M$. Since kE_r is commutative, we might as well consider M^* a left kE_r-module. This action usually does not coincide with the contragredient action of kE_r on the space $\operatorname{Hom}_k(M, k)$ defined via $(x.f)(m) = f(\eta(x).m)$, where η is the antipode of the Hopf algebra kE_r. We denote the module defined via the contragredient action by $\overline{M}$. The module M satisfies the equal images property if and only if $\overline{M}$ satisfies the equal kernels property [11, 7.9]. Moreover, if $M \in \mathrm{CJT}(kE_r)$, then $\overline{M} \in \mathrm{CJT}(kE_r)$ and we have $\mathrm{JType}(M) = \mathrm{JType}(\overline{M})$ [10, 5.2]. These implications still hold if we replace $\overline{M}$ by M^* due to the fact that for $i = 1, \ldots, r$, we have $\eta(x_i) = (p-1)x_i + u_i$, where $u_i \in \operatorname{rad}^2(kE_r)$. Hence $\overline{M}$ satisfies the constant Jordan type, the equal images and the equal kernels property, respectively, if and only if M^* does [11, 1.2, 1.5].

Furthermore, the category $\mathrm{EIP}(kE_r)$ is image-closed [11, 1.10], and dually $\mathrm{EKP}(kE_r)$ is closed under taking submodules.

In most cases, neither the category of modules of constant Jordan type nor the category of modules with the equal images property is tame (cf. [6, 4.5.7] and [16, 4.2.6]). More specifically, $\mathrm{CJT}(kE_r)$ and $\mathrm{EIP}(kE_r)$ are not tame whenever the group algebra kE_r is wild, i.e. if $r \geq 3$ or if $r = 2$ and $p > 2$. Thus, we hope to gain more information about these categories by determining prominent examples of modules with the constant Jordan type and equal images property, respectively.

2.2 Generalized W-modules

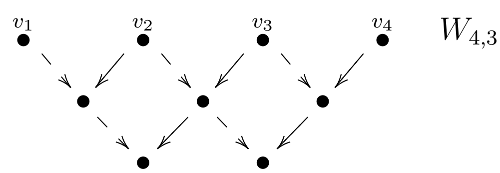
In [11], Carlson, Friedlander and Suslin define a certain class of equal images modules for $\mathbb{Z}_p \times \mathbb{Z}_p$, the so-called W-modules.

Definition 2.6 (cf. [11], 2.1)**.** *Let $n \geq d \geq 1$, $d \leq p$. The kE_2-module $W_{n,d}$ is the module generated by $\{v_1, \ldots, v_n\}$ and relations given by*

$$x_1 v_1 = 0 = x_2 v_n = x_1^d v_n;\;\; x_1^d v_i = 0 = x_2 v_i - x_1 v_{i+1} \text{ for } 1 \leq i \leq n-1.$$

For $n \leq d$, let $W_{n,d} := W_{n,n}$ as above.

The module $W_{4,3}$, for example, can be visualized as follows:



The arrows $\rightarrow$ and $\dashrightarrow$ denote the action of x_1 and x_2, respectively.

These modules play a prominent role in the category $\mathrm{EIP}(kE_2)$:

Proposition 2.7 (cf. [11], 4.1, 4.4)**.** *Let $M \in \mathrm{EIP}(kE_2)$.*

(i) If $\mathrm{rad}^2(M) = 0$, then there exist integers $t \in \mathbb{N}_0$ and $n_1, \ldots, n_t \geq 1$ such that

$$M \cong W_{n_1,2} \oplus \cdots \oplus W_{n_t,2}.$$

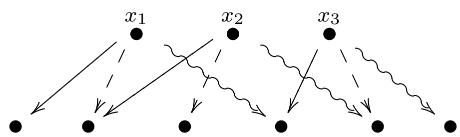
(ii) There is a positive integer n and a surjective homomorphism $W_{n,d} \rightarrow M$, where d satisfies $\mathrm{rad}^d(M) = 0$.

We give a generalization of these $\mathbb{Z}_p \times \mathbb{Z}_p$-modules to elementary abelian p-groups of arbitrary rank. For $n, d \in \mathbb{N},\ d \leq n$, we consider the R-module

$$M_{n,d}^{(r)} := I^{n-d}/I^n,$$

where $I^0 := R$. For $n < d$, we define $M_{n,d}^{(r)} := M_{n,n}^{(r)}$. Since the algebra R is commutative, we can consider the linear dual $W_{n,d}^{(r)} := (M_{n,d}^{(r)})^*$ as a left R-module. Whenever $d \leq p$, the canonical action of R factors through $R/(X_1^p, \ldots, X_r^p)$, so that we can likewise study these modules over kE_r.

The module $M_{3,2}^{(3)}$, for example, can be depicted as follows:



$M_{3,2}^{(3)}$

The dots represent the canonical basis elements given by the monomials in degree one and two and $\rightarrow$, $\dashrightarrow$ and $\rightsquigarrow$ denote the action of x_1, x_2 and x_3, respectively. It is easy to see that in case $r = 2, d \leq p$, the module $W_{n,d}^{(2)}$ is isomorphic to $W_{n,d}$ in Definition 2.6.

Modules of the form $M_{n,d}^{(r)}$ will be referred to as M-**modules** and modules of the form $W_{n,d}^{(r)}$ as W-**modules**, respectively.

Proposition 2.8. *Let* $1 \leq d \leq p$, $n \geq d$. *We have* $M_{n,d}^{(r)} \in \operatorname{EKP}(kE_r)$ *and* $W_{n,d}^{(r)} \in \operatorname{EIP}(kE_r)$.

Proof. Let $(\alpha_1, \ldots, \alpha_r) \in k^r \backslash 0$. We have

$$\ker \left\{ \sum_{i=1}^{r} \alpha_i x_i \colon M_{n,d}^{(r)} \to M_{n,d}^{(r)} \right\} = I^{n-1}/I^n.$$

Hence $M_{n,d}^{(r)}$ satisfies the equal kernels property and, dually, $W_{n,d}^{(r)}$ satisfies the equal images property. $\square$

Some W-modules can be recognized as submodules of the group algebra kE_r, generalizing [11, 2.2].

Proposition 2.9. *For* $d \leq p$, *we have*

$$W_{d,d}^{(r)} \cong \operatorname{rad}^{r(p-1)+1-d}(kE_r).$$

Proof. Observe that kE_r is isomorphic to the restricted enveloping algebra of an r-dimensional abelian Lie algebra with trivial p-map [39, §5] and is equipped with the structure of a Frobenius algebra where the projection $\tau : kE_r \to k$ onto the coefficient of $x_1^{p-1} \cdots x_r^{p-1}$ defines a non-degenerate associative symmetric bilinear-form

$$(.,.) \colon kE_r \times kE_r \to k, (a, b) := \tau(ab),$$

see [7]. Since there is an isomorphism $kE_r/\operatorname{rad}^d(kE_r) \cong M_{d,d}^{(r)}$, the claimed isomorphism of kE_r-modules follows from the associativity of $(.,.)$ together with

$$W_{d,d}^{(r)} \cong (kE_r/\operatorname{rad}^d(kE_r))^* \cong (\operatorname{rad}^d(kE_r))^{\perp} = \operatorname{rad}^{r(p-1)+1-d}(kE_r).$$

$\square$

Observe that the algebraic group $\mathrm{GL}_r(k)$ has an action on the r-dimensional vector space $\bigoplus_{i=1}^r kX_i$ given by $g.X_i = \sum_{s=1}^r g_{si}X_s$ for $1 \leq i \leq r$, $g = (g_{st}) \in \mathrm{GL}_r(k)$. Thereby, $\mathrm{GL}_r(k)$ acts on R and kE_r via automorphisms, leaving $I = (X_1, \dots, X_r) \subseteq R$ and $J = \mathrm{rad}(kE_r)$ invariant. For $M \in \mathrm{mod}\, kE_r$, $g \in \mathrm{GL}_r(k)$, we consider the g**-twist** $M^{(g)}$, where $M^{(g)}$ is the kE_r-module with underlying vector space M and action given by $x.m := (g^{-1}.x)m$. The assignment $M \mapsto M^{(g)}$ defines a functor and thereby an auto-equivalence on $\mathrm{mod}\, kE_r$.

Definition 2.10 (cf. [16], 2)**.** *We call a module* $\mathrm{GL}_r(k)$**-stable** *if there is an isomorphism* $M \cong M^{(g)}$ *for all* $g \in \mathrm{GL}_r(k)$*.*

Since $\mathrm{GL}_r(k)$ acts on $\bigoplus_{i=1}^r kX_i \backslash 0$ with one orbit, $\mathrm{GL}_r(k)$-stable kE_r-modules are necessarily of constant Jordan type. One can easily show that the following holds:

Proposition 2.11. *Let* $M, N \in \mathrm{mod}\, kE_r$ *and* $g \in \mathrm{GL}_r(k)$*.*

(i) We have $\mathrm{Hom}_{kE_r}(M, N) = \mathrm{Hom}_{kE_r}(M^{(g)}, N^{(g)})$*.*

(ii) There is an equality of kE_r*-modules* $\left(M^{(g)}\right)^* = (M^*)^{(g)}$*.*

Proposition 2.12. *Let* $n, d \in \mathbb{N}, d \leq n$*. The* R*-modules* $M_{n,d}^{(r)}$ *and* $W_{n,d}^{(r)}$ *are* $\mathrm{GL}_r(k)$*-stable.*

Proof. Due to Proposition 2.11 (ii), dualizing and twisting are compatible. Hence it suffices to prove the first claim: The module I^{n-d}/I^n is a subfactor of the $\mathrm{GL}_r(k)$-module R and the map

$$\varphi_g \colon I^{n-d}/I^n \to I^{n-d}/I^n, m \mapsto g^{-1}.m$$

defines an isomorphism $M_{n,d}^{(r)} \to (M_{n,d}^{(r)})^{(g)}$ with inverse $\varphi_{g^{-1}}$. □

In the following, we will make use of the graded structures of the algebras R and kE_r and their module categories, respectively.
The algebra $R = \oplus_{i \in \mathbb{Z}^r} R_i$ is a $\mathbb{Z}^r$-graded algebra, where R_i is the k-span of the monomial $X_1^{i_1} \cdots X_r^{i_r}$ for all $i = (i_1, \dots, i_r) \in \mathbb{N}_0^r$ and $R_i = 0$ else. Hence all non-trivial homogeneous components are one-dimensional. For $i \in \mathbb{Z}^r$, we define $|i| = \sum_{j=1}^r i_j$.
Since the ideal $(X_1^p, \dots, X_r^p)$ is homogeneous with respect to this grading, kE_r inherits the $\mathbb{Z}^r$-grading from R. Furthermore, we have $I = \bigoplus_{\substack{i \in \mathbb{Z}^r \\ i \neq 0}} R_i$ and hence $M_{n,d}^{(r)} = I^{n-d}/I^n$ has both as an R- and kE_r-module a canonical $\mathbb{Z}^r$-grading

$$M_{n,d}^{(r)} = \bigoplus_{i \in \mathbb{Z}^r} M_i$$

where M_i is the vector space spanned by $x_1^{i_1} \cdots x_r^{i_r} = X_1^{i_1} \cdots X_r^{i_r} + I^n$ for all $i \in \mathbb{N}_0^r$ such that $n - d \leq |i| \leq n - 1$, and $M_i = 0$ else. Endowed with this grading, $M_{n,d}^{(r)}$ is generated by its components M_i with $i \in \mathbb{N}_0^r$, $|i| = n - d$.

The $\mathbb{Z}^r$-grading induces a $\mathbb{Z}$-grading on the algebra and on the graded modules in a canonical fashion, setting $R_i = \bigoplus_{|(j_1,\dots,j_r)|=i} R_{(j_1,\dots,j_r)}$ and $M_i = \bigoplus_{|(j_1,\dots,j_r)|=i} M_{(j_1,\dots,j_r)}$ for all $i \in \mathbb{Z}$. Note that endowed with this grading, R is a standardly graded algebra.

Theorem 2.13. *For $r \geq 2$ and $n \geq d > 1$, we have an isomorphism of $\mathbb{Z}^r$-graded algebras*

$$\operatorname{End}_R(M_{n,d}^{(r)}) \cong R/I^d \oplus \bigoplus_{\substack{i \in \mathbb{Z}^r \\ |i|=d-1}} k[i]^{s_i}$$

where the right-hand side denotes the trivial extension of R/I^d by a sum of shifts of the trivial R/I^d-bimodule k. In particular, $\operatorname{End}_R(M_{n,d}^{(r)})$ is local and commutative.

Remark 2.14. By computing Hom-spaces, we will moreover show that the s_i are uniquely determined and that they are all equal to zero iff $n = d$.

Proof. We claim that there is a monomorphism

$$\iota\colon R/I^d \to \operatorname{End}_R(M_{n,d}^{(r)})$$

of $\mathbb{Z}^r$-graded k-algebras. Multiplication by an element of R clearly yields an endomorphism of $M_{n,d}^{(r)}$ and we obtain a homomorphism $R \to \operatorname{End}_R(M_{n,d}^{(r)})$ of k-algebras which obviously respects the $\mathbb{Z}^r$-grading. Since $\operatorname{ann}_R(M_{n,d}^{(r)}) = I^d$, the induced morphism ι is injective.

We now show that for $i \in \mathbb{Z}^r$ with $|i| \leq d-2$, the map ι induces an isomorphism of homogeneous components

$$(R/I^d)_i \cong \operatorname{End}_R(M_{n,d}^{(r)})_i \tag{1}$$

Proof of (1): Since $M_{d,d}^{(r)} = R/I^d$, the isomorphism in (1) is obvious for $n = d$. We thus assume $n > d$. Let $\varphi_i \in \operatorname{End}_R(M_{n,d}^{(r)})_i$ and $|i| \leq d-2$. Recall that all non-trivial homogeneous components of $M_{n,d}^{(r)}$ are one-dimensional and the module is generated by its homogeneous components $M_j = (M_{n,d}^{(r)})_j = \left\langle x_1^{j_1} \cdots x_r^{j_r} \right\rangle_k$ with $|j| = n-d$.
For all $1 \leq t \leq r$, we denote by 1_t the element in $\mathbb{N}_0^r$ with the t-th entry being equal to 1 and all others being equal to 0. Given $1 \leq t, t' \leq r$ we denote by $-\mathbb{1}_t + \mathbb{1}_{t'}$ the operation on $\mathcal{K} = \{\kappa \in \mathbb{N}_0^r \mid |\kappa| = n-d\}$ defined via $(-\mathbb{1}_t + \mathbb{1}_{t'})(\kappa) = \kappa - 1_t + 1_{t'}$ if $\kappa_t \neq 0$ and $(-\mathbb{1}_t + \mathbb{1}_{t'})(\kappa) = \kappa$ else. Observe that every non-empty subset of $\mathcal{K}$ that is closed under all such operations is equal to $\mathcal{K}$.
Let us first of all show that $\varphi_i = 0$ if i has a negative entry i_l for some $1 \leq l \leq r$. We know that φ_i certainly vanishes on those M_j, $|j| = n-d$, with $j_l = 0$.
Now assume $\varphi_i(M_k) = 0$, i.e. $\varphi_i(x_1^{k_1} \cdots x_r^{k_r}) = 0$ for some $k \in \mathbb{N}_0^r$, $|k| = n-d$. Let furthermore $t \in \{1, \ldots, r\}$ such that $k_t \neq 0$. For all $t' \in \{1, \ldots, r\}$, we have

$$x_t \varphi_i(x_1^{k_1} \cdots x_r^{k_r} \frac{x_{t'}}{x_t}) = x_{t'} \varphi_i(x_1^{k_1} \cdots x_r^{k_r}). \tag{2}$$

By our assumption, we have $0 = \varphi_i(x_1^{k_1} \cdots x_r^{k_r})$ and hence $x_t \varphi_i(x_1^{k_1} \cdots x_r^{k_r} \frac{x_{t'}}{x_t}) = 0$. Since furthermore $|i| \leq d-2$, this implies $\varphi_i(x_1^{k_1} \cdots x_r^{k_r} \frac{x_{t'}}{x_t}) = 0$ and hence $\varphi_i(M_{k-1_t+1_{t'}}) = 0$. Thus $\{\kappa \in \mathcal{K} \mid \varphi_i(M_\kappa) = 0\}$ is non-empty and closed under operations of the form $-\mathbb{1}_t + \mathbb{1}_{t'}$ and

hence equal to $\mathcal{K}$. We thus obtain $\varphi_i(M_\kappa)=0$ for all $\kappa\in\mathcal{K}$ and hence $\varphi_i=0$.
For $i\in\mathbb{N}_0^r,\ |i|\leq d-2$, we use the fact that non-trivial homogeneous components are one-dimensional and obtain $\varphi_i(x_1^{k_1}\cdots x_r^{k_r})=c_k x_1^{k_1+i_1}\cdots x_r^{k_r+i_r}$ for all $k\in\mathbb{N}_0^r, |k|=n-d$ and scalars c_k. Comparing coefficients in (2) yields that φ_i is multiplication by an element of the form $cx_1^{i_1}\cdots x_r^{i_r}$ with $c\in k$. This proves our claim (1).

In case $i\in\mathbb{N}_0^r,\ |i|=d-1$, we have an isomorphism of vector spaces

$$\mathrm{End}_R(M_{n,d}^{(r)})_i\cong\bigoplus_{\substack{j\in\mathbb{N}_0^r\\ |j|=n-d}}\mathrm{Hom}_k(M_j,M_{i+j})$$

due to the fact that $\varphi(M_j)=0$ for all $\varphi\in\mathrm{End}_R(M_{n,d}^{(r)})_i,\ j\in\mathbb{N}_0^r$ with $|j|>n-d$. Hence

$$\dim_k\mathrm{End}_R(M_{n,d}^{(r)})_i/\iota((R/I^d)_i)=\dim_k\mathrm{End}_R(M_{n,d}^{(r)})_i-\dim_k(R/I^d)_i=|\{j\in\mathbb{N}_0^r\mid |j|=n-d\}|-1.$$

The right-hand term is equal to zero if and only if $n=d$.
For $i\in\mathbb{Z}^r\backslash\mathbb{N}_0^r,\ |i|=d-1$, we have

$$\mathrm{End}_R(M_{n,d}^{(r)})_i\cong\bigoplus_{\substack{j\in\mathbb{N}_0^r\\ |j|=n-d\\ i+j\in\mathbb{N}_0^r}}\mathrm{Hom}_k(M_j,M_{i+j})$$

while $(R/I^d)_i=0$ and the right-hand term being equal to zero if $n=d$. Since furthermore $\mathrm{End}(M_{n,d}^{(r)})_i=0$ for $i\in\mathbb{Z}^r,|i|\geq d$, and maps of degree $d-1$ vanish when composed with maps of degree greater than zero, we obtain the above structure of the endomorphism ring. The algebra $\mathrm{End}_R(M_{n,d}^{(r)})$ is hence local and commutative due to the fact that R/I^d is. □

Corollary 2.15. *Let $1<d\leq n$.*

(i) The R-module $M_{n,d}^{(r)}$ is indecomposable.

(ii) We have $k\cong\mathrm{End}_R(M_{n,d}^{(r)})_0=\mathrm{End}_R^{\mathbb{Z}}(M_{n,d}^{(r)})$.

Remark 2.16. If $d\leq p$, we have an isomorphism of $\mathbb{Z}^r$-graded rings

$$\mathrm{End}_{kE_r}(M_{n,d}^{(r)})\cong\mathrm{End}_R(M_{n,d}^{(r)})$$

whence Corollary 2.15 likewise holds in $\mathrm{mod}\,kE_r$.

In [10, 9.1], the authors pose the question which Jordan types can be realized by modules of constant Jordan type over a given finite group scheme G. This matter has, for example, been studied by Carlson, Friedlander and Pevtsova in [10, 9.] and [9] and by Benson in [6, 4.]. In particular, one is interested in the realization of Jordan types by indecomposables. The Jordan types of the generalized W-modules are as follows:

Proposition 2.17. *For $1 \leq d \leq p$, $n \geq d$, we have*

$$\mathrm{JType}(M_{n,d}^{(r)}) = \binom{r+n-d-1}{n-d}[d] + \sum_{i=1}^{d-1} \binom{r+n-i-2}{n-i}[i] = \mathrm{JType}(W_{n,d}^{(r)})$$

and in particular for $n = d$

$$\mathrm{JType}(M_{n,n}^{(r)}) = [n] + \sum_{i=1}^{n-1} \binom{r+n-i-2}{n-i}[i] = \mathrm{JType}(W_{n,n}^{(r)}).$$

Proof. Let $M = M_{n,d}^{(r)}$. In order to determine $\mathrm{JType}(M)$, we have to compute $\mathrm{JType}(\alpha, M)$ for some p-point α. For $1 \leq i < d$, the number of Jordan blocks of $\alpha(t)_M$ of size $[i]$ equals

$$\begin{aligned}
&2\dim_k \ker \alpha(t)_M^i - \dim_k \ker \alpha(t)_M^{i+1} - \dim_k \ker \alpha(t)_M^{i-1} \\
=&2\dim_k I^{n-i} - \dim_k I^{n-i-1} - \dim_k I^{n-i+1} \\
=&\dim_k R_{n-i} - \dim_k R_{n-i-1} \\
=&\binom{n-i+r-1}{r-1} - \binom{n-i-1+r-1}{r-1} \\
=&\binom{n-i-2+r}{r-2}.
\end{aligned}$$

Since $\alpha(t)^d = 0$, we have $\ker \alpha(t)_M^d = \ker \alpha(t)_M^{d+1}$ and $\ker \alpha(t)^{d-1} = I^{n-d+1}/I^n$. Hence there are $\dim_k R_{n-d} = \binom{n-d+r-1}{n-d}$ blocks of size $[d]$. □

The indecomposability of modules of the form $M_{n,d}^{(2)}$ and $W_{n,d}^{(2)}$, $n \geq d \geq 2$, over the algebra kE_2 follows directly from [11, 4.2], according to which the Jordan type $\sum_{i=1}^{p} a_i[i]$ of a module with the equal images property is such that $a_{i-1} \neq 0$ whenever $a_i \neq 0$, $i \geq 2$. Taking into account that $\mathrm{EIP}(kE_r)$ and $\mathrm{EKP}(kE_r)$ are closed under direct summands and $\mathrm{JType}(W_{n,d}) = (n-d+1)[d] + \sum_{i=1}^{d-1}[i]$, these modules are hence indecomposable if $d \geq 2$. In case $r > 2$, this conclusion does not seem to follow from the computation of Jordan types.

Recall that due to Proposition 2.7 (*i*), in case $r = 2$, the indecomposable equal images modules of Loewy length two are just the modules $W_{n,2}$ with $n \geq 2$. We will show in the following sections that for $r > 2$, the situation is completely different and there is no hope to parametrize the indecomposable equal images modules of Loewy length two in the same fashion. It seems that W-modules are thus not "ubiquitous" in $\mathrm{EIP}(kE_r)$ if $r > 2$.

3 The Beilinson algebra $B(n, r)$

In order to obtain a better understanding of the subcategories of $\operatorname{mod} kE_r$ introduced in the previous chapter, we will now consider the category $\operatorname{mod}_{\mathbb{Z}} kE_r$ of $\mathbb{Z}$-graded modules over the $\mathbb{Z}$-graded algebra kE_r. When studying objects in $\operatorname{mod}_{\mathbb{Z}} kE_r$ that have a bounded support, the generalized Beilinson algebra comes into play. We formulate analogs of the kE_r-module categories we introduced in Chapter 2 as module categories for the Beilinson algebra. Via a homological characterization, we are able to study these categories from the viewpoint of Auslander-Reiten theory, where the categories exhibit interesting properties. Throughout this chapter, let $n, r \geq 2$.

3.1 Definition and basic properties

Definition 3.1. *Let $E(n, r)$ be the path algebra of the quiver $\mathcal{Q}(n, r)$ with n vertices and r arrows between vertices i and $i + 1$ for all $0 \leq i \leq n - 2$.*

$$\begin{tikzcd} 0 \arrow[r, bend left, "\gamma_1^{(0)}"] \arrow[r, bend right, "\gamma_r^{(0)}"'] \arrow[r, phantom, "\vdots"] & 1 \arrow[r, bend left, "\gamma_1^{(1)}"] \arrow[r, bend right, "\gamma_r^{(1)}"'] \arrow[r, phantom, "\vdots"] & 2 & \cdots & n-2 \arrow[r, bend left, "\gamma_1^{(n-2)}"] \arrow[r, bend right, "\gamma_r^{(n-2)}"'] \arrow[r, phantom, "\vdots"] & n-1 \end{tikzcd}$$

The **generalized Beilinson algebra** *$B(n, r)$ is the factor algebra $E(n, r)/\mathcal{I}$ where $\mathcal{I}$ is the ideal generated by the commutativity relations $\gamma_s^{(i)}\gamma_t^{(i-1)} - \gamma_t^{(i)}\gamma_s^{(i-1)}$ for all $s, t \in \{1, \ldots, r\}$ and $i \in \{1, \ldots, n - 2\}$.*

These algebras generalize the algebras of the form $B(n) = B(n, n)$ introduced by Beilinson in [4].

Remark 3.2. The algebraic group $\mathrm{GL}_r(k)$ acts on $E(n, r)$ via automorphisms by defining

$$g.\gamma_j^{(i)} = \sum_{s=1}^{r} g_{sj}\gamma_s^{(i)}$$

and $g.e_i = e_i$ for $g = (g_{st}) \in \mathrm{GL}_r(k)$. Since this action leaves $\mathcal{I}$ invariant, $\mathrm{GL}_r(k)$ acts on $B(n, r)$ via automorphisms and thus on $\operatorname{mod} B(n, r)$ via twisting.

Remark 3.3. Note that we have an antiautomorphism of algebras $\varphi\colon B(n, r) \to B(n, r)$ given by $\varphi(e_i) = e_{n-i-1}$ and $\varphi(\gamma_j^{(i)}) = \gamma_j^{(n-i-2)}$. Since φ is involutory, we have a duality

$$\mathrm{D} = \mathrm{Hom}_k(\varphi^*(-), k)\colon \operatorname{mod} B(n, r) \to \operatorname{mod} B(n, r)$$

with inverse $\mathrm{D} \cong \varphi^*(\mathrm{Hom}_k(-, k))$. The duality functor D is compatible with the Auslander-Reiten translation $\tau = \mathrm{Hom}_k(\mathrm{Tr}(-), k)$ in $\operatorname{mod} B(n, r)$, i.e.

$$\mathrm{D} \circ \tau \cong \tau^{-1} \circ \mathrm{D},$$

due to the fact that $\varphi^* \circ \mathrm{Tr} \cong \mathrm{Tr} \circ \varphi^*$. Moreover, we have $\varphi(g.m) = g.\varphi(m)$ for all $g \in \mathrm{GL}_r(k)$, $m \in \operatorname{mod} B(n, r)$.

We can formulate a modified version of Proposition 2.11.

Proposition 3.4. *Let* $M, N \in \operatorname{mod} B(n,r)$ *and* $g \in \mathrm{GL}_r(k)$.

(i) We have $\operatorname{Hom}_{B(n,r)}(M,N) = \operatorname{Hom}_{B(n,r)}(M^{(g)}, N^{(g)})$.

(ii) There is an equality of $B(n,r)$*-modules* $(\mathrm{D}\, M)^{(g)} = \mathrm{D}\left(M^{(g)}\right)$.

3.2 The functor $\mathfrak{F}_{(n,r)}\colon \operatorname{mod} B(n,r) \to \operatorname{mod}_n kE_r$

Whenever $n \leq p$, we may consider the full subcategory $\mathcal{C}_{[0,n-1]}$ of $\operatorname{mod}_{\mathbb{Z}} kE_r$ containing those objects $M = \bigoplus_{i\in\mathbb{Z}} M_i \in \operatorname{mod}_{\mathbb{Z}} kE_r$ with $\operatorname{supp}(M) \subseteq [0, n-1] := \{0, \ldots, n-1\}$. Hence the essential image of $\mathfrak{F}|_{\mathcal{C}_{[0,n-1]}}$ consists of the $[0, n-1]$-gradable objects in $\operatorname{mod} kE_r$. The following is easy to see (cf. [20, 6.6]):

Remark 3.5. The category $\mathcal{C}_{[0,n-1]}$ is equivalent to $\operatorname{mod} B(n,r)$.

The equivalence between $\mathcal{C}_{[0,n-1]}$ and $\operatorname{mod} B(n,r)$ is such that $M = M_0 \oplus \cdots \oplus M_{n-1} \in \mathcal{C}_{[0,n-1]}$ is a module for $B(n,r)$ where $M_i = e_i M$ for the primitive orthogonal idempotent $e_i \in B(n,r)$ corresponding to the vertex i. The action of x_j on elements in M_i corresponds to the action of $\gamma_j^{(i)}$ on elements in $e_i M$. We will use the notation $M = M_0 \oplus \cdots \oplus M_{n-1}$ both for objects in $\mathcal{C}_{[0,n-1]}$ and for (ungraded) objects in $\operatorname{mod} B(n,r)$.
In the following, we may thus regard $\operatorname{mod} B(n,r)$ as a full subcategory of $\operatorname{mod}_{\mathbb{Z}} kE_r$ and we will see in the next section that we gain a lot by viewing $\mathcal{C}_{[0,n-1]}$ as the module category for a bound quiver algebra.
Let us consider the forgetful functor

$$\mathfrak{F} : \operatorname{mod}_{\mathbb{Z}} kE_r \to \operatorname{mod} kE_r.$$

Restricting $\mathfrak{F}$ to $\operatorname{mod} B(n,r)$ yields a functor

$$\mathfrak{F}_{(n,r)}\colon \operatorname{mod} B(n,r) \to \operatorname{mod}_n kE_r$$

which we will make use of in this chapter. Note that $\mathfrak{F}_{(n,r)}$ is not dense in case $n > 2$. For example, the kE_r-module $M = kE_r/J'$ with $J' = kE_r(x_1^2 - x_2) + J^3$ satisfies $M \in \operatorname{mod}_n kE_r$ for all $n \geq 3$ while M is not contained in the essential images of $\mathfrak{F}$ and $\mathfrak{F}_{(n,r)}$, respectively.

Remark 3.6. Dropping the assumption that $n \leq p$ we may identify $\operatorname{mod} B(n,r)$ with the full subcategory $\mathcal{D}_{[0,n-1]} \subseteq \operatorname{mod}_{\mathbb{Z}} R$ consisting of objects $M \in \operatorname{mod}_{\mathbb{Z}} R$ with $\operatorname{supp}(M) \subseteq [0, n-1]$.

Proposition 3.7. *The algebra* $B(n,r)$ *is a Koszul algebra and its global dimension is* $\operatorname{gldim} B(n,r) = \min\{n-1, r\}$.

Proof. Note that $B(n,r)$ is a standardly graded algebra since the defining relations are homogeneous of degree 2. Recall that the grading is given by the path length grading. We have to determine the graded projective resolutions of the graded simple $B(n,r)$-modules generated in degree 0.

Consider the category $\operatorname{mod}_{\mathbb{Z}} R$. It is well known that the Koszul-complex

$$0 \to R[r]^{\binom{r}{r}} \xrightarrow{d_r} \cdots \xrightarrow{d_2} R[1]^{\binom{r}{1}} \xrightarrow{d_1} R[0] \xrightarrow{d_0} k \to 0$$

provides a minimal projective resolution of the trivial module $k = k[0]$ in $\operatorname{mod}_{\mathbb{Z}} R$. Minimal projective resolutions of $k[i]$, $i \in \mathbb{Z}$, are then given via

$$0 \to R[r+i]^{\binom{r}{r}} \xrightarrow{d_r} \cdots \xrightarrow{d_2} R[1+i]^{\binom{r}{1}} \xrightarrow{d_1} R[i] \xrightarrow{d_0} k[i] \to 0 \tag{3}$$

in $\operatorname{mod}_{\mathbb{Z}} R$. This implies that a projective resolution of $k[i]$, $0 \leq i \leq n-1$, in $\mathcal{D}_{[0,n-1]}$ is given by

$$0 \to \left(R[r+i]^{\binom{r}{r}}\right)_{<n} \xrightarrow{\delta_r} \cdots \xrightarrow{\delta_2} \left(R[1+i]^{\binom{r}{1}}\right)_{<n} \xrightarrow{\delta_1} (R[i])_{<n} \xrightarrow{\delta_0} k[i] \to 0 \tag{4}$$

(cf. [20, 6.6]) with $\delta_j = (d_j)_{<n}$ and where for $1 \leq j \leq r$, we have $(R[j+i])_{<n} = 0$ if $j \geq n-i$. Let $\mathcal{J} = J(B(n,r))$ be the graded radical of $B(n,r)$. For $j < n-i$, we have $\ker \delta_j \subseteq \mathcal{J}\left(R[j+i]^{\binom{r}{j}}\right)_{<n}$ due to the fact that the minimality of the projective resolution (3) forces $\ker d_j \subseteq \mathcal{J}R[j+i]^{\binom{r}{j}}$. Thus the projective resolution (4) is minimal. In view of the identification $\mathcal{D}_{[0,n-1]} \cong \operatorname{mod} B(n,r)$, a minimal projective resolution of $S(i)$ in $\operatorname{mod} B(n,r)$ is hence given by

$$0 \to P(r+i)^{\binom{r}{r}} \to \cdots \to P(1+i)^{\binom{r}{1}} \to P(i) \to S(i) \to 0 \tag{5}$$

if $r < n-i-1$ and by

$$0 \to P(n-1)^{\binom{r}{n-i-1}} \to \cdots \to P(1+i)^{\binom{r}{1}} \to P(i) \to S(i) \to 0 \tag{6}$$

if $r \geq n-i-1$.

Let $\langle - \rangle$ denote the shift in $\operatorname{mod}_{\mathbb{Z}} B(n,r)$. Observe that for $0 \leq j \leq n-2$, we have $\operatorname{Hom}^{\mathbb{Z}}_{B(n,r)}(P(j+1)\langle k\rangle, P(j)) \neq 0$ if and only if $k = 1$. In view of (5) and (6), this implies that $S(i)$ is a Koszul module. Considering the lengths of the projective resolutions (5) and (6), respectively, the projective dimension of $S(i)$ is $\operatorname{pd} S(i) = \min\{n-i-1, r\}$ implying that $\operatorname{gldim} B(n,r) = \min\{n-1, r\}$ is the global dimension of $B(n,r)$.

□

3.3 Module categories for $B(n,r)$

We now define subcategories of $\operatorname{mod} B(n,r)$ that, in case $n \leq p$, correspond to the full subcategories $\mathrm{CR}^j_n(kE_r) \subset \mathrm{CR}^j(kE_r)$, $\mathrm{CJT}_n(kE_r) \subset \mathrm{CJT}(kE_r)$, $\mathrm{EIP}_n(kE_r) \subset \mathrm{EIP}(kE_r)$ as well as

$\mathrm{EKP}_n(kE_r) \subset \mathrm{EKP}(kE_r)$ containing modules of Loewy length at most n.

For $\alpha \in k^r \backslash 0$, we define $\tilde{\alpha} := \sum_{i=0}^{n-2}(\alpha_1\gamma_1^{(i)} + \cdots + \alpha_r\gamma_r^{(i)}) \in B(n,r)$.
Given $M = \bigoplus_{i=0}^{n-1} M_i \in \operatorname{mod} B(n,r)$, left multiplication with $\tilde{\alpha}$ yields a linear operator

$$\alpha_M \colon M \to M$$

such that for $1 \leq j \leq n-1$, $(\alpha_M)^j$ coincides with the left-multiplication with the element $\sum_{i=0}^{n-j-1}((\alpha_1\gamma_1^{(i+j-1)} + \cdots + \alpha_r\gamma_r^{(i+j-1)}) \cdots (\alpha_1\gamma_1^{(i)} + \cdots + \alpha_r\gamma_r^{(i)})) \in B(n,r)$.

Definition 3.8. *We define full subcategories of* $\operatorname{mod} B(n,r)$ *as follows:*

(a) $\mathrm{EIP}(n,r) := \left\{M \in \operatorname{mod} B(n,r) \mid \forall \alpha \in k^r\backslash 0 : \operatorname{im}(\alpha_M) = \bigoplus_{i=1}^{n-1} M_i\right\}$,

(b) $\mathrm{EKP}(n,r) := \{M \in \operatorname{mod} B(n,r) \mid \forall \alpha \in k^r\backslash 0 : \ker(\alpha_M) = M_{n-1}\}$,

(c) $\mathrm{CR}^j(n,r) := \{M \in \operatorname{mod} B(n,r) \mid \exists c_j \in \mathbb{N}_0 \; \forall \alpha \in k^r\backslash 0 : \operatorname{rk}(\alpha_M)^j = c_j\}$,

(d) $\mathrm{CJT}(n,r) := \bigcap_{j=1}^{n} \mathrm{CR}^j(n,r)$.

Moreover, observe that we have $\mathrm{EIP}(n,r) \cup \mathrm{EKP}(n,r) \subset \mathrm{CR}^j(n,r)$ for all $j \geq 1$. Furthermore, note that D restricts to a duality between $\mathrm{EIP}(n,r)$ and $\mathrm{EKP}(n,r)$ (cf. Proposition 3.15 below).

In view of the identification $\operatorname{mod} B(n,r) \cong \mathcal{D}_{[0,n-1]}$, we have:

Proposition 3.9. *Let* $M \in \operatorname{mod} B(n,r)$. *If*

(i) $M \in \mathrm{EIP}(n,r)$, *then* M *is a standardly graded* R*-module and* $M_0 = 0$ *implies* $M = 0$.

(ii) $M \in \mathrm{EKP}(n,r)$, *then* M *is a costandardly graded* R*-module and* $M_{n-1} = 0$ *implies* $M = 0$.

The following shows that the categories defined above in fact correspond to the module categories for kE_r as introduced in Section 3.3.

Proposition 3.10. *In case* $n \leq p$, *the restriction of* $\mathfrak{F}_{(n,r)}$ *to* $\mathcal{X} \in \left\{\mathrm{EIP}(n,r), \mathrm{EKP}(n,r), \mathrm{CR}^j(n,r), \mathrm{CJT}(n,r)\right\}$ *induces a faithful exact functor*

$$\mathfrak{F}_{\mathcal{X}} \colon \mathcal{X} \to \operatorname{mod}_n kE_r$$

such that

(i) for $\mathcal{X} = \mathrm{EIP}(n,r)$, $\mathfrak{F}_{\mathcal{X}}$ *reflects isomorphisms and the essential image consists of the standardly gradable objects in* $\mathrm{EIP}_n(kE_r)$.

(ii) for $\mathcal{X} = \mathrm{EKP}(n,r)$, $\mathfrak{F}_{\mathcal{X}}$ *reflects isomorphisms and the essential image consists of the costandardly gradable objects in* $\mathrm{EKP}_n(kE_r)$.

(iii) for $\mathcal{X} = \mathrm{CR}^j(n,r)$*, the essential image of* $\mathfrak{F}_{\mathcal{X}}$ *consists of the* $[0, n-1]$*-gradable objects in* $\mathrm{CR}^j_n(kE_r)$*.*

Proof. Let $\alpha \in k^r \backslash 0$. For $M \in \mathrm{mod}\, B(n,r)$, the linear operator α_M corresponds to the linear operator $\alpha(t)_{\mathfrak{F}_{(n,r)}(M)}$ on $\mathfrak{F}_{(n,r)}(M)$ given by the p-point α with $\alpha(t) = \alpha_1 x_1 + \cdots + \alpha_r x_r$.
(i): Given $M \in \mathrm{EIP}(n,r)$, we have $\mathrm{im}\, \alpha(t)_{\mathfrak{F}_{(n,r)}(M)} = \bigoplus_{i=1}^{n-1} M_i$. By definition, this implies $\mathfrak{F}_{(n,r)}(M) \in \mathrm{EIP}_n(kE_r)$. For $N \in \mathrm{EIP}(n,r)\backslash 0$, we have $\mathrm{supp}(N) = [0,l]$ for some $0 \leq l \leq n-1$. Thus we have $N[i] \notin \mathrm{EIP}(n,r)$ unless $i = 0$ since $\mathrm{supp}\, N[i] = [i, l+i]$. The functor $\mathfrak{F}_{(n,r)}$ commutes with direct sums and $\mathrm{mod}_n\, kE_r$ is a Krull-Schmidt category. In view of Proposition 1.4, the fibre of an indecomposable object under $\mathfrak{F}_{(n,r)}$ consists of the shifts of an indecomposable object and hence $\mathfrak{F}_{\mathrm{EIP}(n,r)}$ reflects isomorphisms.
Due to Proposition 3.9, $M \in \mathrm{EIP}(n,r)$ is standardly graded. Thus the essential image of $\mathfrak{F}_{\mathrm{EIP}(n,r)}$ consists of standardly gradable objects in $\mathrm{EIP}_n(kE_r)$. Now let $M \in \mathrm{EIP}_n(kE_r)$ be standardly gradable, i.e. $M \cong \mathfrak{F}(\bigoplus_{i\in\mathbb{Z}} M_i)$ for some $\bigoplus_{i\in\mathbb{Z}} M_i \in \mathrm{mod}_{\mathbb{Z}}\, kE_r$. We may assume that $0 = \min \mathrm{supp}(\bigoplus_{i\in\mathbb{Z}} M_i)$ and hence, in view of Remark 2.5, we obtain

$$\mathrm{im}\, \alpha(t)_{\mathfrak{F}(M)} = \mathrm{rad}\, \mathfrak{F}(\bigoplus_{i\in\mathbb{Z}} M_i) = \bigoplus_{i>0} M_i,$$

while the Loewy length of M is bound by n and hence $M_i = 0$ for $i \geq n$. This now implies that $\bigoplus_{i\in\mathbb{Z}} M_i \in \mathrm{EIP}(n,r)$.
(ii): Dual to (i).
(iii): Is clear in view of our general observation above. □

In accordance with the group algebra context, we hence refer to these $B(n,r)$-modules as modules with the equal images property, the equal kernels property and modules with the constant j-rank and constant Jordan type property, respectively.

Proposition 3.11. *Let* $0 \leq i \leq n-2$ *and let* $M = \bigoplus_{j=0}^{n-1} M_j \in \mathrm{EIP}(n,r)$*.*

(i) We have $\mathrm{EIP}(n,r) \cap \mathrm{EKP}(n,r) = (0)$*.*

(ii) If $M_i \neq 0$*, then* $\dim_k M_i > \dim_k M_{i+1}$*.*

Proof. (i): In view of Definition 3.8, it suffices to show that $\mathrm{EIP}(2,r) \cap \mathrm{EKP}(2,r) = (0)$. In view of Proposition 3.10, this can be deduced from the fact that $S(0)$ is the only simple $B(2,r)$-module in $\mathrm{EIP}(2,r)$, $S(1)$ the only simple $B(2,r)$-module in $\mathrm{EKP}(2,r)$ and $\mathrm{EIP}(kE_r) \cap \mathrm{EKP}(kE_r) = \mathrm{add}\, k$ [16, 4.4.3].
(ii): For all $\alpha \in k^r\backslash 0$, we have $\alpha_M(M_i) = M_{i+1}$ and hence $\dim_k M_i \geq \dim_k M_{i+1}$. Assume that $\dim_k M_i = \dim_k M_{i+1}$ and consider the $\mathbb{Z}$-graded R-module $\tilde{M} = (M_{\geq i})_{<i+2}$. Then we have $\tilde{M}[-i] \in \mathcal{D}_{[0,1]}$, with $\tilde{M}[-i] \in \mathrm{EIP}(2,r) \cap \mathrm{EKP}(2,r)$. In view of (i), this implies $\tilde{M}[-i] = 0$, in particular $M_i = 0$.

□

3.4 Homological characterization

In this section, we give a new point of view on our subcategories of $\operatorname{mod} B(n,r)$ which enables us to apply general methods from Auslander-Reiten theory. The approach we present is inspired by work of Happel and Unger [22] on representations of the generalized Kronecker algebra $\mathcal{K}_r$. The authors give a representation $X = (X_1, X_2)$ over $\mathcal{K}_r$ such that the representations $Y = (Y_1, Y_2)$ in the right-perpendicular category $X^\perp$ are exactly those for which the operator $\gamma_1 \colon Y_1 \to Y_2$ corresponding to the arrow γ_1 is bijective [22, 2.1]. It can be seen that the module X arises as the cokernel of the map $\varphi \colon P(1) \to P(0)$ given by $\varphi(e_1) = \gamma_1$.

Recall that $P(i) \cong (R[i])_{<n}$ in $\mathcal{D}_{[0,n-1]} \subseteq \operatorname{mod}_{\mathbb{Z}} R$ for $0 \leq i \leq n-1$. It can be easily seen that for $\alpha \in k^r \backslash 0$ and $0 \leq i \leq n-2$, the map

$$\alpha(i) : P(i+1) \to P(i),\ e_{i+1} \mapsto \alpha_1 \gamma_1^{(i)} + \cdots + \alpha_r \gamma_r^{(i)},$$

i.e. the right multiplication with $\alpha_1 \gamma_1^{(i)} + \cdots + \alpha_r \gamma_r^{(i)}$, defines an embedding of $B(n,r)$-modules. Composition yields embeddings

$$\alpha(i)^j : P(i+j) \to P(i),\ e_{i+j} \mapsto (\alpha_1 \gamma_1^{(i+j-1)} + \cdots + \alpha_r \gamma_r^{(i+j-1)}) \cdots (\alpha_1 \gamma_1^{(i)} + \cdots + \alpha_r \gamma_r^{(i)})$$

for all $0 \leq i \leq n-2,\ 1 \leq j \leq n-i-1$. We let $X_\alpha^{i,j} := \operatorname{coker} \alpha(i)^j = P(i)/\alpha(i)^j(P(i+j))$. For $1 \leq j \leq n-1$, $\alpha \in k^r \backslash 0$, we define

$$X_\alpha^j = \bigoplus_{i=0}^{n-j-1} X_\alpha^{i,j}.$$

In the following, whenever we write Hom or Ext, we refer to the category $\operatorname{mod} B(n,r)$.

Theorem 3.12. *We have*

(a) $\operatorname{EIP}(n,r) = \{ M \in \operatorname{mod} B(n,r) | \forall \alpha \in k^r \backslash 0 : \operatorname{Ext}^1(X_\alpha^1, M) = 0 \}$,

(b) $\operatorname{EKP}(n,r) = \{ M \in \operatorname{mod} B(n,r) | \forall \alpha \in k^r \backslash 0 : \operatorname{Hom}(X_\alpha^1, M) = 0 \}$,

(c) $\operatorname{CR}^j(n,r) = \{ M \in \operatorname{mod} B(n,r) | \exists c_j \in \mathbb{N}_0\ \forall \alpha \in k^r \backslash 0 : \dim_k \operatorname{Ext}^1(X_\alpha^j, M) = c_j \}$
$= \{ M \in \operatorname{mod} B(n,r) | \exists c_j \in \mathbb{N}_0\ \forall \alpha \in k^r \backslash 0 : \dim_k \operatorname{Hom}(X_\alpha^j, M) = c_j \}$.

Proof. Consider the projective resolution $0 \to P(i+j) \xrightarrow{\alpha(i)^j} P(i) \to X_\alpha^{i,j} \to 0$ and, for $M \in \operatorname{mod} B(n,r)$, the exact sequence

$$0 \to \operatorname{Hom}(X_\alpha^{i,j}, M) \to \operatorname{Hom}(P(i), M) \to \operatorname{Hom}(P(i+j), M) \to \operatorname{Ext}^1(X_\alpha^{i,j}, M) \to 0.$$

There is a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(P(i), M) \arrow[r, "\mathrm{Hom}(\alpha(i)^j, M)"] \arrow[d, "\cong"] & \mathrm{Hom}(P(i+j), M) \arrow[d, "\cong"] \\
M_i \arrow[r, "(\alpha_M)^j|_{M_i}"] & M_{i+j}
\end{tikzcd}$$

whence

$$(\alpha_M)^j|_{M_i} : M_i \to M_{i+j}$$

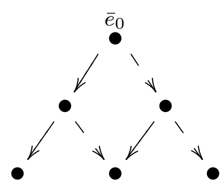
is surjective if and only if $\mathrm{Ext}^1(X_\alpha^{i,j}, M) = 0$ and injective if and only if $\mathrm{Hom}(X_\alpha^{i,j}, M) = 0$. This already yields (a) and (b) and since

$$\mathrm{rk}(\alpha_M)^j = \sum_{i=0}^{n-j-1} (\dim_k M_{i+j} - \dim_k \mathrm{Ext}^1(X_\alpha^{i,j}, M)) = \left(\sum_{i=0}^{n-j-1} \dim_k M_{i+j}\right) - \dim_k \mathrm{Ext}^1(X_\alpha^j, M),$$

we obtain (c). □

Hence we have a homological description of the subcategories defined above that involves a $\mathbb{P}^{r-1}$-family of $B(n,r)$-modules of projective dimension 1. At this juncture, we exploit fundamental homological properties of $\mathrm{mod}\, B(n,r)$ that do not hold in $\mathrm{mod}\, kE_r$.

The module $X^{0,1}_{(0,0,1)} \in \mathrm{mod}\, B(3,3)$ can be visualized as follows



$X^{0,1}_{(0,0,1)}$

where $\to$ denotes the action of $\gamma_1^{(0)}$ and $\gamma_1^{(1)}$ while $\dashrightarrow$ denotes the action of $\gamma_2^{(0)}$ and $\gamma_2^{(1)}$, respectively. Let us list some of the distinctive features of the modules $X_\alpha^{i,j}$. Recall that an indecomposable module with trivial endomorphism ring is referred to as a brick.

Proposition 3.13. *Let $\alpha \in k^r \backslash 0$ and $\iota\colon B(n, r-1) \to B(n,r)$ be the embedding defined via $\gamma_l^k \mapsto \gamma_l^k$ for all $0 \leq k \leq n-2$ and $1 \leq l \leq r-1$.*

(i) We have $\mathrm{pd}_{B(n,r)}(X_\alpha^j) = 1$ for all $1 \leq j \leq n-1$.

(ii) The module $X_\alpha^{i,j}$ is standardly as well as costandardly graded and $\mathrm{supp}(X_\alpha^{i,j}) = [i, n-1]$.

(iii) We have $\dim_k(X_\alpha^{i,j})_i = 1$ and the module $X_\alpha^{i,j}$ is a brick in $\mathrm{mod}\, B(n,r)$.

(iv) All proper submodules of $X_\alpha^{n-2,1}$ are of the form $P(n-1)^{\oplus m}$ for some $m < r$.

(v) *The pullback* $\iota^*(X^{i,1}_{(0,\dots,0,1)})$ *is isomorphic to the projective* $B(n,r-1)$*-module* $\tilde{P}(i)$*, whereas* $\iota^*(\mathrm{D}\,X^{i,1}_{(0,\dots,0,1)}) \cong \tilde{I}(n-i-1)$ *in* $\operatorname{mod} B(n,r-1)$*.*

(vi) *There is an isomorphism* $(X^{i,j}_\alpha)^{(g)} \cong X^{i,j}_{g(\alpha)}$ *for all* $g \in \mathrm{GL}_r(k)$*.*

Moreover, we can compute the Auslander-Reiten translates of modules of the form $X^{i,j}_\alpha$:

Proposition 3.14. *Let* $0 \le i \le n-2,\ 1 \le j \le n-i-1$*. We have*

$$\tau X^{i,j}_\alpha \cong \mathrm{D}\,X^{n-i-j-1,j}_\alpha.$$

Proof. In view of the fact that $\tau \cong \operatorname{Hom}_k(\operatorname{Tr}(-),k)$, it suffices to show that we have an isomorphism $\operatorname{Tr}(X^{i,j}_\alpha) \cong \varphi^*(X^{n-i-j-1,j}_\alpha)$ for $\varphi : B(n,r)^{op} \to B(n,r)$ as in Remark 3.3.
A minimal projective presentation of $X^{i,j}_\alpha$ is given by

$$0 \to P(i+j) \overset{\alpha(i)^j}{\to} P(i) \to X^{i,j}_\alpha \to 0.$$

We obtain

$$\begin{aligned}\operatorname{Tr}(X^{i,j}_\alpha) &\cong \operatorname{coker}\left(\operatorname{Hom}(\alpha(i)^j, B(n,r))\colon \operatorname{Hom}(P(i),B(n,r)) \to \operatorname{Hom}(P(i+j),B(n,r))\right)\\ &\cong \operatorname{coker}\left(\varphi^*(\alpha(n-i-j-1)^j)\colon \varphi^*(P(n-i-1)) \to \varphi^*(P(n-i-j-1))\right)\\ &\cong \varphi^*\left(\operatorname{coker}\left(\alpha(n-i-j-1)^j\colon P(n-i-1) \to P(n-i-j-1)\right)\right)\end{aligned}$$

and hence $\operatorname{Tr}(X^{i,j}_\alpha) \cong \varphi^*(X^{n-i-j-1,j}_\alpha)$.

□

An immediate consequence is the following:

Proposition 3.15. *Let* $M \in \operatorname{mod} B(n,r)$*,* $1 \le j \le n-1$*. Then there is an isomorphism*

$$\operatorname{Ext}^1(X^j_\alpha, \mathrm{D}\,M) \cong \operatorname{Hom}(X^j_\alpha, M).$$

Proof. Since D is a duality, we have $\operatorname{Ext}^1(X^j_\alpha, \mathrm{D}\,M) \cong \operatorname{Ext}^1(M, \mathrm{D}\,X^j_\alpha)$. Due to Proposition 3.14, we have

$$\mathrm{D}\,X^j_\alpha = \mathrm{D}\bigoplus_{i=0}^{n-j-1} X^{i,j}_\alpha = \bigoplus_{i=0}^{n-j-1} \tau X^{i,j}_\alpha = \tau X^j_\alpha.$$

Due to Proposition 3.13 (i), we have $\operatorname{pd}(X^j_\alpha) = 1$ and dually the injective dimension of $\mathrm{D}\,X^j_\alpha$ is bound by one. Hence the Auslander-Reiten formula simplifies to [2, 2.14]

$$\operatorname{Ext}^1(M, \tau X^j_\alpha) \cong \operatorname{Hom}(X^j_\alpha, M),$$

which yields the assertion. □

This indeed verifies that for $M \in \operatorname{mod} B(n,r)$, we have $\mathrm{D}\,M \in \mathrm{EIP}(n,r)$ if and only if $M \in \mathrm{EKP}(n,r)$ while $\mathrm{D}\,M \in \mathrm{CR}^j(n,r)$ if and only if $M \in \mathrm{CR}^j(n,r)$. Thus the notion of constant j-rank is self-dual.

3.5 Modules with the equal images property

In the following, we want to determine the role of the full subcategories defined above from the viewpoint of Auslander-Reiten theory. In particular we are interested in the relative position of these modules in the Auslander-Reiten quiver $\Gamma(n,r)$ of $\operatorname{mod} B(n,r)$. In this section, we are concerned with modules with the equal images property and modules with the equal kernels property while we will turn our attention to the category $\mathrm{CR}^j(n,r)$ in Section 3.7.

Using our homological characterization, we are able to show Theorem (A) stated in the introduction:

Theorem 3.16. *The category* $\mathrm{EIP}(n,r)$ *is the torsion class* $\mathcal{T}$ *of a torsion pair* $(\mathcal{T},\mathcal{F})$ *in* $\operatorname{mod} B(n,r)$ *with* $\mathrm{EKP}(n,r) \subset \mathcal{F}$ *that is closed under the Auslander-Reiten translate* τ *and which contains all preinjective modules. In particular, there are no non-trivial maps* $\mathrm{EIP}(n,r) \to \mathrm{EKP}(n,r)$.
Dually, $\mathrm{EKP}(n,r)$ *is the torsion-free class* $\mathcal{F}'$ *of a torsion pair* $(\mathcal{T}',\mathcal{F}')$ *in* $\operatorname{mod} B(n,r)$ *with* $\mathrm{EIP}(n,r) \subset \mathcal{T}'$ *that is closed under* τ^{-1} *and contains all preprojective modules.*

Proof. Application of Theorem 3.12 directly yields that $\mathrm{EIP}(n,r)$ is extension closed and closed under direct sums. Since $\mathrm{pd}(X_\alpha^j) = 1$ and hence $\mathrm{Ext}^2(X_\alpha^j,-) = 0$, the class is furthermore image closed. Thus $\mathrm{EIP}(n,r)$ is a torsion class in $\operatorname{mod} B(n,r)$.
The corresponding torsion-free objects in $\mathcal{F} = \{M \in \operatorname{mod} B(n,r) | \operatorname{Hom}(\mathcal{T},M) = 0\}$ are those that do not have any non-trivial submodules in $\mathrm{EIP}(n,r)$. In particular, all $N \in \operatorname{mod} B(n,r)$ such that $N_0 = 0$ are torsion-free in view of Proposition 3.9.
We now show that for $M \in \mathrm{EIP}(n,r)$, we have $\tau M \in \mathrm{EIP}(n,r)$. The Auslander-Reiten formula 1.18 yields an isomorphism

$$\mathrm{Ext}^1(X_\alpha^1, \tau M) \cong \left(\underline{\operatorname{Hom}}(M, X_\alpha^1)\right)^*,$$

where

$$\operatorname{Hom}(M, X_\alpha^1) \cong \bigoplus_{i=0}^{n-1} \operatorname{Hom}(M, X_\alpha^{i,1}).$$

For $i \geq 1$, we have $(X_\alpha^{i,1})_0 = 0$ (Prop. 3.13, (ii)) and thus $X_\alpha^{i,1} \in \mathcal{F}$. This yields the isomorphism $\operatorname{Hom}(M, X_\alpha^1) \cong \operatorname{Hom}(M, X_\alpha^{0,1})$. Since $[0] \subset [0, n-1] = \operatorname{supp} X_\alpha^{0,1}$ (Prop. 3.13, (ii)) and by definition $\operatorname{im} \alpha_{X_\alpha^{0,1}} = 0$, we in particular obtain $X_\alpha^{0,1} \notin \mathrm{EIP}(n,r)$.
Due to Proposition 3.13 (ii), (iii), the module $X_\alpha^{0,1}$ is standardly graded and we have $\dim_k (X_\alpha^{0,1})_0 = 1$. Hence every proper submodule $Y \subset X_\alpha^{0,1}$ satisfies $Y_0 = 0$ and thus in view of Proposition 3.9, we have $Y \notin \mathrm{EIP}(n,r)$. This yields $X_\alpha^{0,1} \in \mathcal{F}$ and hence $\operatorname{Hom}(M, X_\alpha^{0,1}) = 0$ which implies $\tau M \in \mathrm{EIP}(n,r)$.
Moreover, Theorem 3.12 directly implies that $\mathrm{EIP}(n,r)$ contains all injective objects in $\operatorname{mod} B(n,r)$ and hence also their τ^m-shifts for all $m \geq 0$, i.e. all preinjectives. The dual statement follows in view of our duality D. Hence $\mathrm{EKP}(n,r)$ is closed under taking submodules and since due to Proposition 3.11 (i), we have $\mathrm{EIP}(n,r) \cap \mathrm{EKP}(n,r) = (0)$, this implies $\mathrm{EKP}(n,r) \subset \mathcal{F}$. $\square$

Definition 3.17. *We denote the torsion-free class associated to* $\mathrm{EIP}(n,r)$ *by* $\mathcal{F}(n,r)$ *and the corresponding torsion radical by* $\mathfrak{t}_{(n,r)}$.

Remark 3.18. The inclusion $\mathrm{EKP}(n,r) \subset \mathcal{F}(n,r)$ is proper: We have $X_\alpha^{0,1} \in \mathcal{F}(n,r)$ as shown in the proof of Theorem 3.16 while in view of Theorem 3.12, we have $X_\alpha^{0,1} \notin \mathrm{EKP}(n,r)$.

Theorem 3.16 implies that a mesh in the Auslander-Reiten quiver $\Gamma(n,r)$ of $\operatorname{mod} B(n,r)$

$$\begin{tikzcd}
 & {[E_1]} \arrow[dr] & \\
{[\tau M]} \arrow[ur] \arrow[dr] & \vdots & {[M]} \\
 & {[E_t]} \arrow[ur] &
\end{tikzcd}$$

with M in $\mathrm{EIP}(n,r)$ is completely contained in $\mathrm{EIP}(n,r)$. We thus obtain

Corollary 3.19. *Let* $M \in \mathrm{EIP}(n,r)$ *be indecomposable. Then* $(\to M) \subseteq \mathrm{EIP}(n,r)$. *Dually, for* $M \in \mathrm{EKP}(n,r)$, *we have* $(M \to) \subseteq \mathrm{EKP}(n,r)$.

While in general not much is known about the shape of the connected components of the Auslander-Reiten quiver $\Gamma(n,r)$, we are able to make more precise statements for $\mathbb{Z}A_\infty$-components of $\Gamma(n,r)$. Recall that modules in the bottom row of such components are referred to as quasi-simple.

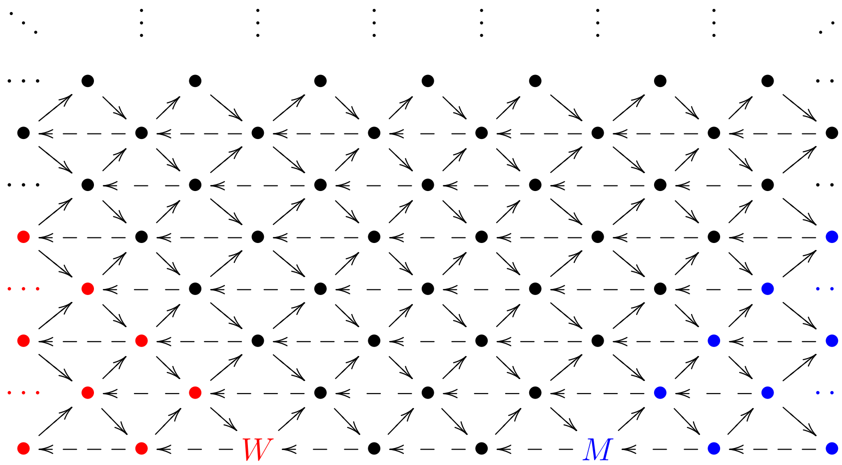
Proposition 3.20. *Let* $\mathcal{C}$ *be a regular* $\mathbb{Z}A_\infty$*-component of* $\Gamma(n,r)$.

(i) If $\mathrm{EIP}(n,r) \cap \mathcal{C} \neq \emptyset$, *then either* $\mathcal{C} \subseteq \mathrm{EIP}(n,r)$ *or there exists a quasi-simple module* $W = W_\mathcal{C}$ *such that* $(\to W_\mathcal{C}) = \mathcal{C} \cap \mathrm{EIP}(n,r)$.

(ii) Dually, if $\mathrm{EKP}(n,r) \cap \mathcal{C} \neq \emptyset$, *then either* $\mathcal{C} \subseteq \mathrm{EKP}(n,r)$ *or there exists a quasi-simple module* $M = M_\mathcal{C}$ *such that* $(M_\mathcal{C} \to) = \mathcal{C} \cap \mathrm{EKP}(n,r)$.

Proof. Since in every regular $\mathbb{Z}A_\infty$-component the irreducible maps from top to bottom are surjective, $\mathrm{EIP}(n,r) \cap \mathcal{C} \neq \emptyset$ yields the existence of a quasi-simple module W in $\mathcal{C}$ that belongs to $\mathrm{EIP}(n,r)$. If all quasi-simple modules belong to $\mathrm{EIP}(n,r)$, Corollary 3.19 yields $\mathcal{C} \subset \mathrm{EIP}(n,r)$. In view of Corollary 3.19 and the fact that any two quasi-simple modules are successor, resp. predecessor of one another, we can choose k maximal such that $W_\mathcal{C} := \tau^{-k}(W) \in \mathrm{EIP}(n,r)$ and $(\to W_\mathcal{C}) = \mathcal{C} \cap \mathrm{EIP}(n,r)$. Dual properties of modules in $\mathrm{EKP}(n,r)$ yield the assertion. □

Hence, whenever a regular $\mathbb{Z}A_\infty$- component $\mathcal{C}$ of $\Gamma(n,r)$ contains both objects from $\mathrm{EIP}(n,r)$ and $\mathrm{EKP}(n,r)$, these components have the following shape:



The red and blue bullets indicate that the corresponding module is an object in $\mathrm{EIP}(n,r)$, respectively in $\mathrm{EKP}(n,r)$. We may thus define the following invariant:

Definition 3.21. *Let $\mathcal{C}$ be a regular $\mathbb{Z}A_\infty$-component of $\Gamma(n,r)$ with $W_\mathcal{C}, M_\mathcal{C} \in \mathcal{C}$ such that*

$$(\to W_\mathcal{C}) = \mathcal{C} \cap \mathrm{EIP}(n,r) \text{ and } (M_\mathcal{C} \to) = \mathcal{C} \cap \mathrm{EKP}(2,r).$$

Then the **width** *$\mathcal{W}(\mathcal{C})$ measures the size of the gap between the modules $W_\mathcal{C}$ and $M_\mathcal{C}$, i.e. is determined by the property*

$$\tau^{\mathcal{W}(\mathcal{C})+1}(M_\mathcal{C}) = W_\mathcal{C}.$$

This invariant provides information concerning the properties of modules in the region between the two cones as we will see in Section 3.7.

3.6 W-modules are modules for the Beilinson algebra

We are concerned with the special role that W- and M- modules play as modules for generalized Beilinson algebras.
Recall that for all $1 \leq d \leq n$, $m \geq d$, the $\mathbb{Z}$-graded module $M_{m,d}^{(r)}$ inherits a $\mathbb{Z}$-grading from the polynomial ring R, where $\operatorname{supp}(M_{m,d}^{(r)}) = [m-d, m-1]$ in $\operatorname{mod}_\mathbb{Z} R$. Hence we have $M_{m,d}^{(r)}[n-m] \in \mathcal{D}_{[0,n-1]}$. It is easily seen that $M_{m,d}^{(r)}[n-m]$ is an object in $\mathrm{EKP}(n,r)$, since for all $\alpha \in k^r \backslash 0$, we have $\ker \alpha_{M_{m,d}^{(r)}[n-m]} = (I^{m-1}/I^m)[n-m] = (M_{m,d}^{(r)}[n-m])_{n-1}$.

Likewise, the $\mathbb{Z}$-grading on W-modules in $\operatorname{mod}_\mathbb{Z} R$ is such that $\operatorname{supp}(W_{m,d}^{(r)}) = [-m+1, -m+d]$ and hence $W_{m,d}^{(r)}[m-1] \in \mathcal{D}_{[0,n-1]}$ is an object in $\mathrm{EIP}(n,r)$. For our duality D on $\operatorname{mod} B(n,r)$, we have

$$\mathrm{D}\, M_{m,d}^{(r)}[n-m] \cong W_{m,d}^{(r)}[m-1].$$

Note furthermore that for $1 \leq d \leq n$, we have isomorphisms $M_{d,d}^{(r)}[n-d] \cong P(n-d)$ and $W_{d,d}^{(r)}[d-1] \cong I(d-1)$ in $\operatorname{mod} B(n,r)$. Since for $d \geq 2$, $M_{m,d}^{(r)}$ is a brick in $\operatorname{mod}_\mathbb{Z} R$ by Corollary

2.15, $M_{m,d}^{(r)}[n-m]$ is a brick in $\operatorname{mod} B(n,r)$, i.e. $\operatorname{End}_{B(n,r)}(M_{m,d}^{(r)}[n-m]) \cong k$.

In the remainder of this thesis, we are concerned with $B(n,r)$-modules and hence shorten notation and write $M_{m,d}^{(r)}$ for the $B(n,r)$-module $M_{m,d}^{(r)}[n-m]$ and likewise $W_{m,d}^{(r)}$ for the $B(n,r)$-module $W_{m,d}^{(r)}[m-1]$. Summing up the above, we have:

Proposition 3.22. *Let $2 \leq d \leq n$, $m \geq d$. The $B(n,r)$-module*

(i) $M_{m,d}^{(r)} \in \operatorname{EKP}(n,r)$ *is a brick with* $\operatorname{supp} M_{m,d}^{(r)} = [n-d, n-1]$.

(ii) $W_{m,d}^{(r)} \in \operatorname{EIP}(n,r)$ *is a brick with* $\operatorname{supp} W_{m,d}^{(r)} = [0, d-1]$.

Moreover, we have:

Lemma 3.23. *Let $m \geq n$.*

(i) We have $\operatorname{rad}^i P(0) = M_{n,n-i}^{(r)}$ *for $0 \leq i < n$.*

(ii) Let $0 \leq i < n$. Then $\operatorname{Ext}^1_{B(n,r)}(M_{m,n}^{(r)}, S(i)) \neq 0$ *iff $i = 1$.*

Proof. (i): obvious.
(ii): We want to show that $\operatorname{Hom}(\Omega(M_{m,n}^{(r)}), S(i)) \cong \operatorname{Ext}^1(M_{m,n}^{(r)}, S(i)) \neq 0$ iff $i = 1$. Due to Proposition 3.7, $B(m,r)$ is a Koszul algebra with grading given by the path length grading. Denote by $\mathcal{J} = J(B(m,r))$ the graded radical of $B(m,r)$. Let furthermore $\mathcal{F}\colon \operatorname{mod}_{\mathbb{Z}} B(m,r) \to \operatorname{mod} B(m,r)$ denote the forgetful functor and let $\langle - \rangle$ be the shift in $\operatorname{mod}_{\mathbb{Z}} B(m,r)$. Observe that for $M \in \operatorname{mod}_{\mathbb{Z}} B(m,r)$, we have $\mathcal{F}(\mathcal{J}M) \cong \operatorname{rad} \mathcal{F}(M)$.
The projective module $P(0) = M_{m,m}^{(r)}$ generated in degree 0 trivially is a Koszul module in $\operatorname{mod}_{\mathbb{Z}} B(m,r)$. Due to Proposition 1.8, this implies that $M := \mathcal{J}^{m-n}(M_{m,m}^{(r)})\langle n-m \rangle$ is a Koszul module in $\operatorname{mod}_{\mathbb{Z}} B(m,r)$ while in view of (i), we have $M_{m,n}^{(r)} \cong \mathcal{F}(M)$. Hence a minimal graded projective resolution of M in $\operatorname{mod}_{\mathbb{Z}} B(m,r)$ is of the form

$$\cdots \to P^2 \xrightarrow{\delta_2} P^1 \xrightarrow{\delta_1} P(M) \to M \to 0,$$

where the P^i are generated in degree i and the δ_i are maps of degree 0.
Recall that $\operatorname{mod} B(m,r) \cong \mathcal{D}_{[0,m-1]}$. We have $\operatorname{supp} \mathcal{F}(M) = [m-n, m-1]$, and hence we can consider the resolution above as a minimal graded projective resolution of M in $\operatorname{mod}_{\mathbb{Z}} B(n,r) \hookrightarrow \operatorname{mod}_{\mathbb{Z}} B(m,r)$.
In $\operatorname{mod} B(n,r) \cong \mathcal{D}_{[0,n-1]} \subset \operatorname{mod}_{\mathbb{Z}} R$, the standardly R-graded module $\mathcal{F}(M)$ satisfies $\operatorname{supp} \mathcal{F}(M) = [0, n-1]$ and hence the projective cover $P(M) \in \operatorname{mod}_{\mathbb{Z}} B(n,r)$ of M is of the form $P(0)^{\oplus s}$, generated in degree 0. Now $\delta_1 \in \operatorname{Hom}(P^1, P(0)^{\oplus s})_0$ together with the fact that $P^1 \in \operatorname{mod}_{\mathbb{Z}} B(n,r)$ is generated in degree 1 implies that $P^1 \cong P(1)\langle 1 \rangle^{\oplus t}$ for some $t \in \mathbb{N}$. This yields the assertion, since a minimal projective resolution of $M_{m,n}^{(r)} \in \operatorname{mod} B(n,r)$ is given by

$$\cdots \to \mathcal{F}(P^2) \xrightarrow{\delta_2} \mathcal{F}(P^1) \xrightarrow{\delta_1} \mathcal{F}(P(M)) \to M_{m,n}^{(r)} \to 0,$$

while $\mathcal{F}(P^1) \cong P(1)^{\oplus t}$. $\square$

We will make use of the foregoing lemma in Chapter 5, where we are concerned with spaces of morphisms between generalized M- and generalized W-modules.

A distinctive property of sincere generalized M- and W-modules is that their τ^k-shifts are objects in $\mathrm{EIP}(n,r)$ for $k > 0$ and objects in $\mathrm{EKP}(n,r)$ for $k < 0$, which follows from Theorem 3.16 together with:

Theorem 3.24. *Let $r \geq 3$, $m > n$. Then $\tau M^{(r)}_{m,n} \in \mathrm{EIP}(n,r)$ and $\tau^{-1} W^{(r)}_{m,n} \in \mathrm{EKP}(n,r)$.*

Proof. We want to apply Theorem 3.12 again in combination with the Auslander-Reiten formula and thus show that for all $\alpha \in k^r \setminus 0$, there are only trivial maps $M^{(r)}_{m,n} \to X^1_\alpha$. Since $M^{(r)}_{m,n}$ is generated by $(M^{(r)}_{m,n})_0$, we have $\mathrm{Hom}(M^{(r)}_{m,n}, X^1_\alpha) \cong \mathrm{Hom}(M^{(r)}_{m,n}, X^{0,1}_\alpha)$. By Proposition 2.12, $M^{(r)}_{m,n}$ is $\mathrm{GL}_r(k)$-stable. Now let $g \in \mathrm{GL}_r(k)$ such that $g(\alpha) = (0,\ldots,0,1)$. Due to Proposition 3.13, we have $(X^{0,1}_\alpha)^{(g)} \cong X^{0,1}_{(0,\ldots,0,1)}$. Since by Proposition 3.4

$$\mathrm{Hom}(M^{(r)}_{m,n}, X^{0,1}_\alpha) = \mathrm{Hom}((M^{(r)}_{m,n})^{(g)}, (X^{0,1}_\alpha)^{(g)}) \cong \mathrm{Hom}(M^{(r)}_{m,n}, X^{0,1}_{(0,\ldots,0,1)}),$$

we may hence assume that $\alpha = (0,0,\ldots,1)$.
Assume that there exists a non-trivial map $\varphi\colon M^{(r)}_{m,n} \to X^{0,1}_{(0,\ldots,0,1)}$. Then φ is surjective due to Proposition 3.13, $(ii),(iii)$. We have $\gamma^{(i)}_r \in \mathrm{ann}_{B(n,r)} X^{0,1}_\alpha$ and thus $\gamma^{(i)}_r M^{(r)}_{m,n} \subseteq \ker\varphi$ for all $0 \leq i \leq n-2$. Note that $N := \sum_{i=0}^{n-2} \gamma^{(i)}_r M^{(r)}_{m,n}$ is a submodule of $M^{(r)}_{m,n}$ such that $\gamma^{(i)}_r$ acts trivially on $\tilde{M} := M^{(r)}_{m,n}/N$. Observe that for the embedding ι from Proposition 3.13, we have $\iota^*(\tilde{M}) \cong M^{(r-1)}_{m,n} \oplus \tilde{S}(0)^{\oplus t}$ for some $t \in \mathbb{N}$. Moreover, Proposition 3.13 (v) yields $\iota^*(X^{0,1}_\alpha) \cong \tilde{P}(0)$ for the projective indecomposable $B(n,r-1)$-module corresponding to the vertex 0.
Thus there results a split epimorphism $M^{(r-1)}_{m,n} \oplus \tilde{S}(0)^{\oplus t} \to \tilde{P}(0)$ of $B(n,r-1)$-modules which is a contradiction since by Theorem 2.15, $M^{(r-1)}_{m,n}$ is indecomposable and furthermore not isomorphic to $\tilde{P}(0)$ in $\mathrm{mod}\, B(n,r-1)$ due to the fact that $m > n$ and $r > 2$. Hence we have $\tau M^{(r)}_{m,n} \in \mathrm{EIP}(n,r)$. Our duality D on $\mathrm{mod}\, B(n,r)$ now yields the assertion since $\mathrm{D}\,\tau M^{(r)}_{m,n} \cong \tau^{-1} W^{(r)}_{m,n}$. □

Note that the above theorem does not hold in case $r = 2$. Since modules of the form $M^{(2)}_{m,2}$ are preprojective, we have $\tau M^{(2)}_{m,2} \in \mathrm{EKP}(2,2)\setminus 0$ for $m > 2$.

We will show that generalized W-modules as in Theorem 3.24 determine $\mathbb{Z}A_\infty$-components of $\Gamma(n,r)$ as in Definitions 3.21 with $\mathcal{W}(\mathcal{C}) = 0$. We prove this for the case $n = 2$ in Chapter 4 and by using the theory of one-point extensions, we generalize this result to arbitrary $n \geq 3$ in Chapter 5.

Farnsteiner has shown in [16, 5.2] that the distribution of kE_2-modules with the equal images property in the stable Auslander-Reiten quiver $\Gamma_s(G)$ of kE_2 depends on their Loewy length. He considers the $\mathbb{Z}A_\infty$-component $\Theta_{n,d}$ of $\Gamma_s(G)$ containing the quasi-simple module $W^{(2)}_{n,d}$

and shows that $\mathrm{EIP}(kE_2) \cap \Theta_{n,d}$ is finite if $d \leq p-2$ or if $d = p-1$ and $n \geq p$. In all other cases, $\Theta_{n,d}$ contains a non-empty cone consisting of modules with the equal images property and in which the quasi-simple modules are W-modules.

3.7 Modules of constant j-rank

We now want to study the class of modules of constant j-rank over the Beilinson algebra $B(n,r)$. Friedlander and Pevtsova have shown that for the group algebra kE_r, the constant j-rank property is in fact a property of the components of the stable Auslander-Reiten quiver of kE_r [18, 4.7]. We will see that the situation is rather different in our context.
In contrast to the categories $\mathrm{EIP}(n,r)$ and $\mathrm{EKP}(n,r)$, the category $\mathrm{CR}^j(n,r)$ is neither closed under arbitrary extensions nor under images or submodules, respectively:

Let $\alpha \in k^r \backslash 0$ and consider the module $X_\alpha^{0,1} \in B(2,r)$. Then there exist exact sequences

$$0 \to P(1) \to P(0) \to X_\alpha^{0,1} \to 0 \tag{7}$$

and

$$0 \to S(1)^{\oplus r-1} \to X_\alpha^{0,1} \to S(0) \to 0 \tag{8}$$

where $P(1), P(0), S(1), S(0) \in \mathrm{CR}^1(2,r)$ while $X_\alpha^{0,1} \notin \mathrm{CR}^1(2,r)$ due to the fact that $\operatorname{rk} \alpha(t)_{X_\alpha^{0,1}} = 0$ and $\operatorname{rk} \beta(t)_{X_\alpha^{0,1}} = 1$ for $[\beta] \neq [\alpha] \in \mathbb{P}^{r-1}$.

Hence in view of the sequence (7), $\mathrm{CR}^j(n,r)$ is not closed under images in general and, dually, not under submodules. As can be seen from (8), $\mathrm{CR}^j(n,r)$ is not closed under extensions in general. We are, however, able to make specific statements about the category $\mathrm{CR}^j(n,r)$ concerning images and extensions.

Lemma 3.25. *Let $0 \to A \to B \to C \to 0$ be an exact sequence in* $\operatorname{mod} B(n,r)$.

(i) If $A \in \mathrm{EIP}(n,r)$, then $B \in \mathrm{CR}^j(n,r)$ if and only if $C \in \mathrm{CR}^j(n,r)$.

(ii) If $C \in \mathrm{EKP}(n,r)$, then $B \in \mathrm{CR}^j(n,r)$ if and only if $A \in \mathrm{CR}^j(n,r)$.

Proof. We show (i), Statement (ii) follows dually. Let $A \in \mathrm{EIP}(n,r)$. Since for all $\alpha \in k^r \backslash 0$, we have $\mathrm{Ext}^2(X_\alpha^j, -) = 0$, there is an exact sequence

$$\mathrm{Ext}^1(X_\alpha^j, A) \to \mathrm{Ext}^1(X_\alpha^j, B) \to \mathrm{Ext}^1(X_\alpha^j, C) \to 0,$$

where $\mathrm{Ext}^1(X_\alpha^j, A) = 0$ since $A \in \mathrm{EIP}(n,r)$. Thus the dimension of the rightmost term does not depend on α iff the dimension of the middle term does not. □

Let us now consider Auslander-Reiten sequences in $\operatorname{mod} B(n,r)$.

Lemma 3.26. *Let $M \in \operatorname{mod} B(n,r)$ be indecomposable and and let*

$$0 \to \tau M \to E \to M \to 0$$

be the Auslander-Reiten sequence ending in M. If two out of the three modules are of constant j-rank, so is the third.

Proof. Let $\alpha \in k^r \backslash 0$. In view of Proposition 3.14, we can rule out the case that $M \cong X_\alpha^{i,j}$ for some $i \leq n-j-1$ since both $X_\alpha^{i,j}$ and $\tau X_\alpha^{i,j} \cong \mathrm{D}\, X_\alpha^{n-i-j-1,j}$ are not of constant j-rank. Due to the fact that $E \to M$ is right almost split and M is not a direct summand of X_α^j, any morphism $X_\alpha^j \to M$ factors through E. Hence we get the following exact sequence

$$0 \to \operatorname{Hom}(X_\alpha^j, \tau M) \to \operatorname{Hom}(X_\alpha^j, E) \to \operatorname{Hom}(X_\alpha^j, M) \to 0$$

and the assertion follows with Theorem 3.12. □

A direct consequence of Lemma 3.26 is the following:

Proposition 3.27. *Let $\mathcal{C}$ be a component as in Definition 3.21.*

(i) If all quasi-simple modules in $\mathcal{C}$ are of constant j-rank, then $\mathcal{C} \subseteq \mathrm{CR}^j(n,r)$.

(ii) In particular, if $\mathcal{W}(\mathcal{C}) = 0$, then $\mathcal{C} \subseteq \mathrm{CJT}(n,r)$.

In order to make statements about the occurrence of modules with the constant-j-rank property in $\mathbb{Z}A_\infty$-components as in Definition 3.21 with $\mathcal{W}(\mathcal{C}) > 0$, we prove the following:

Proposition 3.28. *Let $\mathcal{C}$ be a component as in Definition 3.21 and let $1 \leq k \leq l$. Then*

(i) $[k]\tau^{-k} W_\mathcal{C} \in \mathrm{CR}^j(n,r)$ iff $[l]\tau^{-k} W_\mathcal{C} \in \mathrm{CR}^j(n,r)$.

(ii) $\tau^k M_\mathcal{C}(k) \in \mathrm{CR}^j(n,r)$ iff $\tau^k M_\mathcal{C}(l) \in \mathrm{CR}^j(n,r)$.

Proof. We show (i), (ii) follows dually. It suffices to show that given $l' \geq k$, we have $[l']\tau^{-k} W_\mathcal{C} \in \mathrm{CR}^j(n,r)$ if and only if $[l'+1]\tau^{-k} W_\mathcal{C} \in \mathrm{CR}^j(n,r)$. The quasi-socle $\tau^{-k+l'} W_\mathcal{C}$ of $[l'+1]\tau^{-k} W_\mathcal{C}$ satisfies the equal images property since $-k + l' \geq 0$ and we have a short exact sequence (cf. [35, 2.2])

$$0 \to \tau^{-k+l'} W_\mathcal{C} \to [l'+1]\tau^{-k} W_\mathcal{C} \to [l']\tau^{-k} W_\mathcal{C} \to 0.$$

In view of Lemma 3.25, $[l']\tau^{-k}(W_\mathcal{C}) \in \mathrm{CR}^j(n,r)$ if and only if $[l'+1]\tau^{-k} W_\mathcal{C} \in \mathrm{CR}^j(n,r)$. □

Corollary 3.29. *Let $\mathcal{C}$ be a component as in Definition 3.21 with $\mathcal{W}(\mathcal{C}) = 1$. Then either $\mathcal{C} \subseteq \mathrm{CR}^j(n,r)$ or there are no indecomposable modules of constant j-rank in $\mathcal{C}$ apart from the modules in $\mathrm{EIP}(n,r) \cap \mathcal{C}$ and $\mathrm{EKP}(n,r) \cap \mathcal{C}$.*

Proof. Consider the module $[1]\tau^{-1}W_{\mathcal{C}} = \tau M_{\mathcal{C}}(1)$. Then due to Proposition 3.28 (ii), $\tau M_{\mathcal{C}}(1)$ has constant j-rank, if and only if $\tau M_{\mathcal{C}}(l) = [l]\tau^{-l}W_{\mathcal{C}}$ has constant j-rank for all $l \geq 1$. Since

$$\left\{[k]\tau^{-l}W_{\mathcal{C}} | k \geq l \geq 1\right\} = \{M \in \mathcal{C} | M \notin \mathrm{EIP}(n,r) \cup \mathrm{EKP}(n,r)\},$$

the assertion follows with Proposition 3.28 (i). $\square$

The following statement is concerned with Auslander-Reiten sequences that correspond to the torsion-theoretic sequences given in Proposition 1.12.

Proposition 3.30. *Let $0 \to A \to B \to C \to 0$ be an Auslander-Reiten sequence in $\operatorname{mod} B(n,r)$ such that A is in $\mathrm{EIP}(n,r)$ and C is in $\mathcal{F}(n,r)$. Then B is indecomposable. If furthermore $C \in \mathrm{EKP}(n,r)$, then $B \in \mathrm{CJT}(n,r) \backslash (\mathrm{EIP}(n,r) \cup \mathrm{EKP}(n,r))$.*

Proof. Assume that there exists a decomposition $B = \oplus_{i \in I} B_i$ such that $|I| \geq 2$. Then for reasons of dimension it is not possible that all irreducible maps $A \to B_i$ are injective and all irreducible maps $B_j \to C$ are surjective. Thus there exists an epimorphism $A \to B_i$ or a monomorphism $B_i \to C$ for some $i \in I$. This now implies that B_i satisfies the equal images property or that B is torsion-free, respectively. In case $B_i \in \mathrm{EIP}(n,r)$, every morphism $B_i \to C$ is trivial in view of Theorem 3.16. With the same argument $B_i \in \mathcal{F}(n,r)$ yields that every morphism $A \to B_i$ is trivial, a contradiction. Thus B is indecomposable and furthermore $B \notin \mathrm{EIP}(n,r)$ and $B \notin \mathcal{F}(n,r)$. If C satisfies the equal kernels property, then by Lemma 3.25, we have $C \in \mathrm{CJT}(n,r)$. $\square$

We will make use of Proposition 3.30 in Lemma 3.36 below to prove that certain torsion-free modules determine $\mathbb{Z}A_\infty$-components.

3.8 Restrictions to $B(k,r)$, $k < n$

In this section, we introduce two types of restriction functors which will provide us with a deeper understanding of the categories $\mathrm{EIP}(n,r)$ and $\mathrm{EKP}(n,r)$. We will, moreover, make use of these restriction functors in combination with torsion-theoretic arguments to show that certain torsion-free indecomposable $B(n,r)$-modules determine $\mathbb{Z}A_\infty$-components of $\Gamma(n,r)$.

Let $2 \leq k < n$. Define

$$\varphi_n^k \colon \mathcal{Q}(k,r) \to \mathcal{Q}(n,r)$$

via $\varphi_n^k(j) = j$ for $j \in \mathcal{Q}(k,r)_0 = \{0,\dots,k-1\}$ and $\varphi_n^k(\hat{\gamma}_s^{(i)}) = \gamma_s^{(i)}$ for $1 \leq i \leq k-1$. We can, for example, visualize φ_3^2 as follows

$$0 \underset{\hat{\gamma}_r^{(0)}}{\overset{\hat{\gamma}_1^{(0)}}{\vdots}} 1 \quad \xrightarrow{\varphi_3^2} \quad 0 \underset{\gamma_r^{(0)}}{\overset{\gamma_1^{(0)}}{\vdots}} 1 \underset{\gamma_r^{(1)}}{\overset{\gamma_1^{(1)}}{\vdots}} 2$$

Moreover, define

$$\psi_n^k \colon \mathcal{Q}(k,r) \to \mathcal{Q}(n,r)$$

via $\psi_n^k(j) = j+n-k$ for $j \in \mathcal{Q}(k,r)_0 = \{0,\dots,k-1\}$ and $\psi_n^k(\hat{\gamma}_s^{(i)}) = \gamma_s^{(n+i-k)}$ for $1 \leq i \leq k-1$. We can visualize ψ_3^2 as follows

$$\begin{tikzcd} 0 \arrow[r, bend left, dashed, "\hat{\gamma}_1^{(0)}"] \arrow[r, bend right, dashed, "\hat{\gamma}_r^{(0)}"'] \arrow[r, phantom, "\vdots"] & 1 & \arrow[r, "\psi_3^2"] & {} & 0 \arrow[r, bend left, "\gamma_1^{(0)}"] \arrow[r, bend right, "\gamma_r^{(0)}"'] \arrow[r, phantom, "\vdots"] & 1 \arrow[r, bend left, dashed, "\gamma_1^{(1)}"] \arrow[r, bend right, dashed, "\gamma_r^{(1)}"'] \arrow[r, phantom, "\vdots"] & 2 \end{tikzcd}$$

There result functors

$$\Phi_n^k : \operatorname{mod} B(n,r) \to \operatorname{mod} B(k,r)$$

and

$$\Psi_n^k : \operatorname{mod} B(n,r) \to \operatorname{mod} B(k,r)$$

with $\Phi_n^k(M) = \sum_{i=0}^{k-1} e_i M$ and $\Psi_n^k(M) = \sum_{i=n-k}^{n-1} e_i M$ and action given via pullback along φ_n^k and ψ_n^k, respectively.

Remark 3.31. Note that $\mathrm{D} \circ \Psi_n^k \cong \Phi_n^k \circ \mathrm{D}$, where D denotes the duality on $\operatorname{mod} B(n,r)$ and $\operatorname{mod} B(k,r)$, respectively.

The following is a direct consequence of the definition of the categories $\mathrm{EIP}(n,r)$ and $\mathrm{EKP}(n,r)$, respectively:

Proposition 3.32. *Let $M \in \operatorname{mod} B(n,r)$ and let $2 \leq k < n$.*

(i) If $M \in \mathrm{EIP}(n,r)$, then $\Phi_n^k(M) \in \mathrm{EIP}(k,r)$ and $\Psi_n^k(M) \in \mathrm{EIP}(k,r)$.

(ii) If $M \in \mathrm{EKP}(n,r)$, then $\Phi_n^k(M) \in \mathrm{EKP}(k,r)$ and $\Psi_n^k(M) \in \mathrm{EKP}(k,r)$.

(iii) If $\Phi_n^k(M) \in \mathrm{EIP}(k,r)$ and $\Psi_n^{n-k+1}(M) \in \mathrm{EIP}(n-k+1,r)$, then $M \in \mathrm{EIP}(n,r)$.

(iv) If $\Phi_n^k(M) \in \mathrm{EKP}(k,r)$ and $\Psi_n^{n-k+1}(M) \in \mathrm{EKP}(n-k+1,r)$, then $M \in \mathrm{EKP}(n,r)$.

Recall that due to Theorem 3.16, we have $\tau M \in \mathrm{EIP}(n,r)$ whenever an indecomposable module $M \in \operatorname{mod} B(n,r)$ satisfies the equal images property. In view of Proposition 3.32 (i), the following is a generalization of this property:

Proposition 3.33. *Let $M \in \operatorname{mod} B(n,r)$.*

(i) We have $\tau M \in \mathrm{EIP}(n,r)$ if $\Psi_n^2(M) \in \mathrm{EIP}(2,r)$.

(ii) We have $\tau^{-1} M \in \mathrm{EKP}(n,r)$ if $\Phi_n^2(M) \in \mathrm{EKP}(2,r)$.

Proof. We prove (i), (ii) is dual. In view of Theorem 3.12, we need to show that

$$\underline{\mathrm{Hom}}(M, X^1_\alpha) \cong \mathrm{Ext}^1(X^1_\alpha, \tau M) = 0$$

for all $\alpha \in k^r\backslash 0$. Since for $1 \leq i \leq n-2$, $X^{i,1}_\alpha$ embeds into $X^{i-1,1}_\alpha$, it suffices to show

$$\mathrm{Hom}(M, X^{0,1}_\alpha) = 0.$$

We first of all claim that there exists a monomorphism

$$\eta\colon X^{0,1}_\alpha \to (\mathrm{D}\, X^{0,1}_\alpha)^{\oplus s}$$

for some $s \in \mathbb{N}$. In view of Proposition 3.4, dualizing and twisting are compatible and furthermore we have

$$\mathrm{Hom}(X^{0,1}_\alpha, (\mathrm{D}\, X^{0,1}_\alpha)^{\oplus s}) = \mathrm{Hom}((X^{0,1}_\alpha)^{(g)}, ((\mathrm{D}\, X^{0,1}_\alpha)^{\oplus s})^{(g)})$$

for all $g \in \mathrm{GL}_r(k)$. Moreover, there is $g \in \mathrm{GL}_r(k)$ such that $g(\alpha) = (0, \ldots, 0, 1)$. In view of Proposition 3.13, this implies $(X^{0,1}_\alpha)^{(g)} \cong X^{0,1}_{(0,\ldots,0,1)}$ and hence we may assume that $\alpha = (0, \ldots, 0, 1)$. Now consider the embedding $\iota\colon B(n, r-1) \to B(n, r)$ defined via $\gamma^k_l \mapsto \gamma^k_l$ for all $0 \leq k \leq n-1$ and $1 \leq l \leq r-1$ as given in Proposition 3.13. We have $\iota^*(X^{0,1}_\alpha) \cong \tilde{P}(0)$ and $\iota^*(\mathrm{D}\, X^{0,1}_\alpha) \cong \tilde{I}(n-1)$. Since $\mathrm{soc}_{B(n,r-1)}\, \tilde{P}(0) \cong \tilde{S}(n-1)^{\oplus s}$ for some $s \in \mathbb{N}$, we have an injective hull

$$\hat{\eta} : \iota^*(X^{0,1}_\alpha) \to \iota^*(\mathrm{D}\, X^{0,1}_\alpha)^{\oplus s}$$

of $B(n, r-1)$-modules. Since the arrows $\gamma^{(i)}_r$ annihilate both $X^{0,1}_\alpha$ and $\mathrm{D}\, X^{0,1}_\alpha$, $\hat{\eta}$ is in fact also $B(n, r)$-linear, i.e. we have $\hat{\eta} = \iota^*(\eta)$ for some monomorphism $\eta \in \mathrm{Hom}_{B(n,r)}(X^{0,1}_\alpha, (\mathrm{D}\, X^{0,1}_\alpha)^{\oplus s})$.

Assume now that $\mathrm{Hom}(M, X^{0,1}_\alpha) \neq 0$ and let $\delta \in \mathrm{Hom}(M, X^{0,1}_\alpha)\backslash 0$. Since $\mathrm{Hom}(M, -)$ is left exact, we thus obtain $0 \neq \mathrm{Hom}(M, \eta)(\delta) \in \mathrm{Hom}(M, (\mathrm{D}\, X^{0,1}_\alpha)^{\oplus s})$. Thus there exists a non-trivial map $\rho : M \to \mathrm{D}\, X^{0,1}_\alpha$. In view of Proposition 3.13, we have $\mathrm{soc}\, \mathrm{D}\, X^{0,1}_\alpha = (\mathrm{D}\, X^{0,1}_\alpha)_{n-1}$ which yields

$$\Psi^2_n(\rho) \in \mathrm{Hom}_{B(2,r)}(\Psi^2_n(M), \Psi^2_n(\mathrm{D}\, X^{0,1}_\alpha))\backslash 0.$$

On the other hand, we have $\Psi^2_n(M) \in \mathrm{EIP}(2, r)$, which in view of Remark 3.31 implies $\Phi^2_n(\mathrm{D}\, M) \cong \mathrm{D}\, \Psi^2_n(M) \in \mathrm{EKP}(2, r)$ and hence

$$\begin{aligned}
\mathrm{Hom}_{B(2,r)}(\Psi^2_n(M), \Psi^2_n(\mathrm{D}\, X^{0,1}_\alpha)) &\cong \mathrm{Hom}_{B(2,r)}(\mathrm{D}\, \Psi^2_n(\mathrm{D}\, X^{0,1}_\alpha), \mathrm{D}\, \Psi^2_n(M)) \\
&\cong \mathrm{Hom}_{B(2,r)}(\Phi^2_n(X^{0,1}_\alpha), \Phi^2_n(\mathrm{D}\, M)) \\
&= 0
\end{aligned}$$

since $\Phi^2_n(X^{0,1}_\alpha)$ is isomorphic to the module $\hat{X}^{0,1}_\alpha$ defined in $B(2, r)$. Thus we have a contradiction and the assertion follows. □

Using these restriction functors, we are able to make statements about the Auslander-Reiten translates of modules of the form $X^{i,j}_\alpha$:

Proposition 3.34. *Let $0 \leq i \leq n-2,\ 1 \leq j \leq n-i-1$. If $r > 2$, then $\tau^2 X_\alpha^{i,j} \in \operatorname{EIP}(n,r)$.*

Proof. Due to Proposition 3.14, there is an isomorphism $\tau^2(X_\alpha^{i,j}) \cong \tau \operatorname{D} X_\alpha^{n-i-j-1,j}$. In view of Proposition 3.33, we have $\tau \operatorname{D} X_\alpha^{n-i-j-1,j} \in \operatorname{EIP}(n,r)$, whenever $i+j < n-1$, since in that case $(\operatorname{D} X_\alpha^{n-i-j-1,j})_{n-1} = (X_\alpha^{n-i-j-1,j})_0 = 0$ and hence $\Psi_n^2(\operatorname{D} X_\alpha^{n-i-j-1,j}) \in \operatorname{EIP}(2,r)$.
For $i+j = n-1$, we want to show that $\operatorname{Hom}(\operatorname{D} X_\alpha^{0,j}, X_\beta^1) = 0$ for all $\beta \in k^r \backslash 0$, since in view of the Auslander-Reiten formula, this implies $\operatorname{Ext}^1(X_\beta^1, \tau \operatorname{D} X_\alpha^{0,j}) = 0$. Due to Proposition 3.13 the module $\operatorname{D} X_\alpha^{0,j}$ is standardly graded with $\operatorname{supp} \operatorname{D} X_\alpha^{0,j} = [0, n-1]$. Hence we have $\operatorname{Hom}(\operatorname{D} X_\alpha^{0,j}, X_\beta^1) \cong \operatorname{Hom}(\operatorname{D} X_\alpha^{0,j}, X_\beta^{0,1})$. Now assume that $\varphi \in \operatorname{Hom}(\operatorname{D} X_\alpha^{0,j}, X_\beta^{0,1}) \backslash 0$. Both $\operatorname{D} X_\alpha^{0,j}$ and $X_\beta^{0,1}$ are standardly graded with $0 = \min \operatorname{supp} \operatorname{D} X_\alpha^{0,j} = \min \operatorname{supp} X_\beta^{0,1}$ and we have $\dim_k (X_\beta^{0,1})_0 = 1$, which yields that φ is surjective. Since $r > 2$, we have $\dim_k (X_\beta^{0,1})_{n-1} > 1$ while $\dim_k (\operatorname{D} X_\alpha^{0,j})_{n-1} = \dim_k (X_\alpha^{0,j})_0 = 1$. This contradicts the surjectivity of φ. Hence the assertion holds. $\square$

Using torsion theoretic arguments, we will show that certain modules determine $\mathbb{Z}A_\infty$-components in $\Gamma(n,r)$. A main tool is provided via the following lemma:

Lemma 3.35. *Let $2 \leq k < n$ and let*

$$\Sigma: \quad 0 \to \tau M \xrightarrow{f} E \xrightarrow{g} M \to 0$$

be an Auslander-Reiten sequence in $\operatorname{mod} B(n,r)$.

(i) If $\operatorname{Hom}(P(t), M) \neq 0$ for some $0 \leq t \leq n-k-1$, then the restriction of Σ via Ψ_n^k splits.

(ii) If $\operatorname{Hom}(P(t), \tau M) \neq 0$ for some $k \leq t \leq n-1$, then the restriction of Σ via Φ_n^k splits.

Proof. We prove (i), (ii) is dual. Since $e_t.M \neq 0$ for some $t \leq n-k-1$, we have a proper embedding

$$\zeta \colon M_{\geq n-k} \to M$$

of $B(n,r)$-modules. Since Σ is almost split, we thus obtain a map $h \colon M_{\geq n-k} \to E$ such that $\zeta = g \circ h$. This implies $\operatorname{id}_{\Psi_n^k(M)} = \Psi_n^k(\zeta) = \Psi_n^k(g) \circ \Psi_n^k(h)$ and hence the restriction of Σ via Ψ_n^k splits. $\square$

Proposition 3.36. *Let $n \geq 3, r \geq 2$. Assume that $C \in \mathcal{F}(n,r)$ is indecomposable and there exist $2 \leq k \leq t < n$ such that*

(a) $\Phi_n^k(C) \in \mathcal{F}(k,r)$ is indecomposable,

(b) $\Psi_n^{n-k+1}(C) \in \operatorname{EIP}(n-k+1,r)$

(c) $\operatorname{Hom}(P(t), \tau C) \neq 0$.

Then C is a quasi-simple module in a $\mathbb{Z}A_\infty$-component of $\Gamma(n,r)$.

Proof. We are going to show by induction that there is an infinite chain of irreducible epimorphisms

$$\cdots \to [n]C \to [n-1]C \to \cdots \to [1]C = C$$

such that the middle-term of the Auslander-Reiten sequence ending in $[i]C$ has two indecomposable direct summands if $i > 1$ and is indecomposable if $i = 1$.

Let $\Phi = \Phi_n^k$ and $\Psi = \Psi_n^{n-k+1}$. Moreover, define $[1]C := C$ and let $i \geq 1$. Assume that $[1]C, \ldots, [i]C$ are indecomposable objects in $\operatorname{mod} B(n,r)$ that satisfy the following properties

(i) $\operatorname{Hom}(P(t), \tau[j]C) \neq 0$ for $1 \leq j \leq i$.

(ii) $\Psi([j]C) \in \operatorname{EIP}(n-k+1, r)$ for $1 \leq j \leq i$.

(iii) If $i \geq 2$, there is an Auslander-Reiten sequence

$$\Sigma_1 : 0 \to \tau[1]C \to [2]C \to [1]C \to 0$$

and for $2 < j \leq i$, there are Auslander-Reiten sequences

$$\Sigma_{j-1} : 0 \to \tau[j-1]C \xrightarrow{\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}} \tau([j-2]C) \oplus [j]C \xrightarrow{\begin{pmatrix} g_1 & g_2 \end{pmatrix}} [j-1]C \to 0$$

where f_1, g_2 are epimorphisms and f_2, g_1 are monomorphisms.

(iv) There are isomorphisms

$$\operatorname{coker}(\tau[j-1]C \to [j]C) \cong [1]C$$

and $\Phi(\tau[j-1]C) \oplus \Phi([1]C) \cong \Phi([j]C)$ for $2 \leq j \leq i$.

Let $[i+1]C$ be the module that completes the Auslander-Reiten sequence

$$\boxed{\Sigma_i : 0 \to \tau[i]C \to M_i \oplus [i+1]C \to [i]C \to 0}$$

with $M_i = 0$ if $i = 1$ and $M_i = \tau[i-1]C$ else.
We prove

$$\boxed{\text{The module } [i+1]C \text{ is indecomposable and (i)-(iv) hold for } j = i+1.}$$

Let us first of all show that $[i+1]C$ is indecomposable. For $i = 1$, due to (b) and Proposition 3.33, we have $\tau C \in \operatorname{EIP}(n,r)$ and hence Proposition 3.30 implies that $[2]C$ is indecomposable.

Thus we may assume $i > 1$.
Due to (i) and Lemma 3.35, Σ_i splits after restriction via Φ, i.e.

$$\Phi(\tau[i-1]C) \oplus \Phi([i+1]C) \cong \Phi(\tau[i]C) \oplus \Phi([i]C).$$

Due to (iv), we have

$$\Phi(\tau[i-1]C) \oplus \Phi([1]C) \cong \Phi([i]C)$$

and hence

$$\Phi([i+1]C) \cong \Phi(\tau[i]C) \oplus \Phi([1]C).$$

Assume that X'' is an indecomposable direct summand of $[i+1]C$ with the property that $\Phi([1]C)$ is a direct summand of $\Phi(X'')$.
Now assume that $X' \neq 0$ is another indecomposable direct summand of $[i+1]C$. Due to (ii) and Proposition 3.33, we have $\tau[j]C \in \mathrm{EIP}(n,r)$, in particular $\Phi(\tau[j]C) \in \mathrm{EIP}(k,r)$ and $\Psi(\tau[j]C) \in \mathrm{EIP}(n-k+1,r)$ for all $1 \leq j \leq i$. Hence we obtain $\Phi(X') \in \mathrm{EIP}(k,r)$ since $\Phi(X')$ is a direct summand of $\Phi(\tau[i]C)$. In view of the restriction of the sequence Σ_i via Ψ, we have $\Psi([i+1]) \in \mathrm{EIP}(n-k+1,r)$ due to (ii) and the fact that $\Psi(\tau[i]C) \in \mathrm{EIP}(n-k+1,r)$. This yields $\Psi(X') \in \mathrm{EIP}(n-k+1,r)$. In view of Proposition 3.32 (iii), we obtain $X' \in \mathrm{EIP}(n,r)$.
Assume that the irreducible map $f : X' \to [i]C$ is surjective. This implies $[i]C \in \mathrm{EIP}(n,r)$ and in view of (iii), we have $[1]C \in \mathrm{EIP}(n,r)$ due to the fact that $\mathrm{EIP}(n,r)$ is closed under images. Since $[1]C$ is torsion-free, this is a contradiction. Hence f must be injective. Due to (iv), the module $\mathfrak{t}_{(n,r)}([i]C) \cong \tau[i-1]C$ is the unique maximal torsion submodule of $[i]C$, whence f factors through the indecomposable module $\tau[i-1]C$. In view of the irreducibility of the maps f and $\tau[i-1]C \to [i]C$, this implies that the indecomposable modules X' and $\tau[i-1]C$ are isomorphic. Since Σ_i and Σ_{i-1} are Auslander-Reiten sequences, this implies

$$2 \leq \dim_k \mathrm{Irr}(\tau[i-1]C, [i]C) = \dim_k \mathrm{Irr}([i]C, [i-1]C) = 1,$$

a contradiction. Thus $[i+1]C \cong X''$ is indecomposable.

Let us now show that the properties (i)-(iv) hold for $[i+1]C$. Due to the fact that property (iii) is satisfied for $j = i$, we obtain that f_1 is surjective and g_1 is injective, while for reasons of dimension this implies that g_2 is surjective and f_2 is injective. Hence property (iii) holds for $j = i+1$. In particular, there is an epimorphism $\tau[i+1]C \to \tau[i]C$ and since (i) holds for $j = i$, it must hold for $j = i+1$ by the projectivity of $P(t)$. In view of Σ_i, (ii) holds for $j = i+1$ since $\Psi([i]C),\ \Psi(\tau[i]C) \in \mathrm{EIP}(n-k+1,r)$. Condition (iv) holds for $j = i+1$ since there is an isomorphism

$$[1]C \cong \mathrm{coker}(\tau[j-1]C \to [j]C) \cong \mathrm{coker}(\tau[j]C \to [j+1]C)$$

by exactness of the sequence Σ_i.

Hence, since $[1]C$ satisfies (i) and (ii), we are thus able to recursively define modules $[j]C$ for all $j \geq 1$ that satisfy the above conditions.

Let $\mathcal{C}$ be the component of $\Gamma(n,r)$ containing C and let $\mathcal{D} = \{\tau^k[j]C \mid j \in \mathbb{N},\ k \in \mathbb{Z}\} \subseteq \mathcal{C}$. Then $\mathcal{D}$ is τ-stable as well as τ^{-1}-stable and closed under predecessors and successors and $\mathcal{C}\backslash\mathcal{D}$ has the same properties. Since $\mathcal{C}$ is connected we have $\mathcal{C} = \mathcal{D}$. Moreover, for all $j \in \mathbb{N}$, we have $\tau^k[j]C \in \mathrm{EIP}(n,r)$ if and only if $k > 0$ due to the fact that $\tau[j]C \in \mathrm{EIP}(n,r)$ while $[1]C \notin \mathrm{EIP}(n,r)$ and the fact that $\mathrm{EIP}(n,r)$ is closed under images as well as under τ. This implies that $\mathcal{C}$ is non-periodic and hence a $\mathbb{Z}A_\infty$-component. □

Under certain restrictions, the requirement (c) is automatically satisfied and for $r = 2$, we can show the following:

Proposition 3.37. *Let $n \geq 3$. Assume that $C \in \mathcal{F}(n,2)$ is indecomposable and there exists $2 \leq k < n$ such that*

(a) $\Phi_n^k(C) \in \mathrm{EKP}(k,2)$ is indecomposable,

(b) $\Psi_n^{n-k+1}(C) \in \mathrm{EIP}(n-k+1,2)$

Then C is a quasi-simple module in a $\mathbb{Z}A_\infty$-component $\mathcal{C}$ of $\Gamma(n,2)$ and $\mathcal{W}(\mathcal{C}) = 1$.

Proof. It suffices to show that $\mathrm{Ext}^1(C, S(k)) \neq 0$ since in view of the Auslander-Reiten formula, we obtain $\overline{\mathrm{Hom}}(S(k), \tau C) \neq 0$, in particular $\mathrm{Hom}(P(k), \tau C) \neq 0$, whence we can apply Proposition 3.36. We have an isomorphism

$$\mathrm{Ext}^1(C, S(k)) \cong \mathrm{Ext}^1(\mathrm{D}\,S(k), \mathrm{D}\,C) \cong \mathrm{Ext}^1(S(n-k-1), \mathrm{D}\,C),$$

while $\Phi_n^{n-k+1}(\mathrm{D}\,C) \cong \mathrm{D}\,\Psi_n^{n-k+1}(C) \in \mathrm{EKP}(n-k+1,2)$ and $\Psi_n^k(\mathrm{D}\,C) \cong \mathrm{D}\,\Phi_n^k(C) \in \mathrm{EIP}(k,2)$.

Define $i := n-k-1$. A projective resolution of $S(i)$ is given by

$$0 \to P(i+2) \overset{\delta_{i+1}}{\to} P(i+1)^{\oplus 2} \overset{\delta_i}{\to} P(i) \to S(i) \to 0$$

where δ_{i+1} is given by right-multiplication with $\begin{pmatrix} -\gamma_2^{(i)} & \gamma_1^{(i)} \end{pmatrix}$ and δ_i is given by right-multiplication with $\begin{pmatrix} \gamma_1^{(i)} \\ \gamma_2^{(i)} \end{pmatrix}$ (cf. the proof of Proposition 3.7).
Applying the functor $\mathrm{Hom}(-, \mathrm{D}\,C)$ we obtain a complex

$$0 \to \mathrm{Hom}(P(i), \mathrm{D}\,C) \overset{\delta_i^*}{\to} \mathrm{Hom}(P(i+1)^{\oplus 2}, \mathrm{D}\,C) \overset{\delta_{i+1}^*}{\to} \mathrm{Hom}(P(i+2)), \mathrm{D}\,C) \to \cdots .$$

Since $\Psi_n^{n-i-1}(\mathrm{D}\,C) = \Psi_n^k(\mathrm{D}\,C) \in \mathrm{EIP}(k,2)$, the linear map

$$\delta_{i+1}^* : (\mathrm{D}\,C)_{i+1}^{\oplus 2} \to (\mathrm{D}\,C)_{i+2},\ (x,y) \mapsto -\gamma_2^{(i)}x + \gamma_1^{(i)}y$$

is surjective and moreover $\dim_k(\mathrm{D}\,C)_{i+1} > \dim_k(\mathrm{D}\,C)_{i+2}$ in view of Proposition 3.11. On the other hand, the map

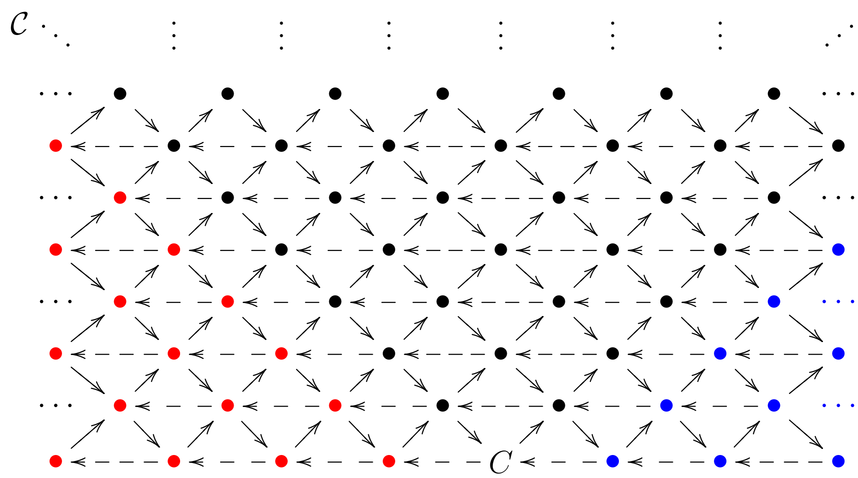
$$\delta_i^* : (\mathrm{D}\,C)_i \to (\mathrm{D}\,C)_{i+1}^{\oplus 2},\ x \mapsto (\gamma_1^{(i)}x, \gamma_2^{(i)}x)$$

is injective. In view of Proposition 3.11, we have $\dim_k(\mathrm{D}\,C)_{i+1} > \dim_k(\mathrm{D}\,C)_i$ due to the fact that $\Phi_n^{i+2}(\mathrm{D}\,C) = \Phi_n^{n-k+1}(\mathrm{D}\,C) \in \mathrm{EKP}(n-k+1,2)$. Hence

$$\dim_k(\operatorname{im}\delta_i^*) = \dim_k(\mathrm{D}\,C)_i < \dim_k(\mathrm{D}\,C)_{i+1} < 2\dim_k(\mathrm{D}\,C)_{i+1} - \dim_k(\mathrm{D}\,C)_{i+2} = \dim_k \ker \delta_{i+1}^*$$

whence $\mathrm{Ext}^1(S(n-k-1), \mathrm{D}\,C) \neq 0$. By Proposition 3.36, C is hence quasi-simple in a $\mathbb{Z}A_\infty$-component $\mathcal{C}$ of $\Gamma(n,2)$. We have $C \notin \mathrm{EIP}(n,2)$ and $C \notin \mathrm{EKP}(n,2)$ in view of the assumption and Proposition 3.11 together with Proposition 3.32. Due to Proposition 3.33, we have $\tau C \in \mathrm{EIP}(n,2)$ and $\tau^{-1}C \in \mathrm{EKP}(n,2)$ and hence $\mathcal{W}(\mathcal{C}) = 1$. □

These components can be visualized as follows:



In view of Corollary 3.29, the module C has constant j-rank if and only if all modules in $\mathcal{C}$ have constant j-rank.
The following shows that each indecomposable object in $\mathrm{EKP}(n-1,2)$ “induces” a $\mathbb{Z}A_\infty$-component of $\Gamma(n,2)$:

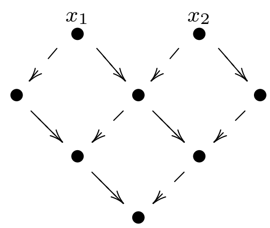
Corollary 3.38. *If $C \in \operatorname{mod} B(n,2)$ is indecomposable such that $\mathrm{Hom}(P(n-1), C) = 0$ and $\Phi_n^{n-1}(C) \in \mathrm{EKP}(n-1,2)$, then C is a quasi-simple module in a $\mathbb{Z}A_\infty$-component $\mathcal{C}$ of $\Gamma(n,2)$ with $\mathcal{W}(\mathcal{C}) = 1$ and $\mathcal{C} \subseteq \mathrm{CJT}(n,2)$.*

Proof. For $k = n-1$ the conditions for Proposition 3.37 are satisfied. Moreover, since $\Phi_n^{n-1}(C) \in \mathrm{EKP}(n-1,2)$ and $e_{n-1}.C = 0$, we have $C \in \mathrm{CJT}(n,2)$. In view of Corollary 3.29, this implies $\mathcal{C} \subseteq \mathrm{CJT}(n,2)$. □

Let us now consider sincere modules, i.e. modules $M \in \operatorname{mod} B(n,r)$ such that $M_i \neq 0$ for all $0 \leq i \leq n-1$, that satisfy the properties in Proposition 3.37.
Let $I = (X_1, X_2) \subseteq R = k[X_1, X_2]$ and let $m \geq 2$. Consider the R-module

$$C_{d,e}^{(m)} = I^{m-d}/((X_1^m, X_2^m) + I^{m+e})$$

where $2 \leq d \leq m$ and $1 \leq e \leq m-1$. These modules generalize the kE_2-modules of the form $C_{2,1}^{(m)}$ which were considered by Carlson, Friedlander and Pevtsova in [10, 2.4] and by Benson in [6, 4.1.6]. The module $C_{2,2}^{(3)}$, for example, can be depicted as follows:



$C_{2,2}^{(3)}$

Remark 3.39. Let $m \geq 2$ and let $2 \leq d \leq m$, $1 \leq e \leq m-1$.

(i) We have $C_{d,e}^{(m)} \in \operatorname{mod}_{\mathbb{Z}} R$ with $\operatorname{supp} C_{d,e}^{(m)} = [m-d, m+e-1]$.

(ii) We have $C_{d,e}^{(m)}[d-m] \in \operatorname{mod} B(d+e,2)$.

When considering the module $C_{d,e}^{(m)}[d-m] \in \operatorname{mod}_{\mathbb{Z}} R$ as a module in $\operatorname{mod} B(d+e,2)$, we shorten notation and write $C_{d,e}^{(m)} \in \operatorname{mod} B(d+e,2)$. The module $C_{d,e}^{(m)}$ is indecomposable due to the fact that $\Phi_{d+e}^2(C_{d,e}^{(m)}) \cong M_{m,2}^{(2)} \in \mathrm{EKP}(d,2)$ is indecomposable and $C_{d,e}^{(m)}$ is generated by $(C_{d,e}^{(m)})_0$.

Proposition 3.40. *Let $\mathcal{C}$ be the component of $\Gamma(d+e,2)$ containing the module $C_{d,e}^{(m)}$. Then $\mathcal{C}$ is of type $\mathbb{Z}A_\infty$, $C_{d,e}^{(m)}$ is quasi-simple in $\mathcal{C}$ and $\mathcal{W}(\mathcal{C}) = 1$.*

Proof. The module $\Phi_{d+e}^d(C_{d,e}^{(m)}) \cong M_{m,d}^{(2)} \in \mathrm{EKP}(d,2)$ is indecomposable and furthermore, we have $\Psi_{d+e}^{e+1}(C_{d,e}^{(m)}) \cong W_{m,e+1}^{(2)} \in \mathrm{EIP}(e+1,2)$. The fact that $\Phi_{d+e}^d(C_{d,e}^{(m)}) \in \mathrm{EKP}(d,2)$ particularly implies $C_{d,e}^{(m)} \in \mathcal{F}(d+e,2)$ since a submodule $M \subseteq C_{d,e}^{(m)}$ with $M \in \mathrm{EIP}(d+e,2)\backslash 0$ necessarily satisfies $0 \neq \Phi_{d+e}^d(M) \in \mathrm{EIP}(d,2)$ in view of Proposition 3.9. Application of Proposition 3.37 yields that $C_{d,e}^{(m)}$ is quasi-simple in the $\mathbb{Z}A_\infty$-component $\mathcal{C}$ and we have $\mathcal{W}(\mathcal{C}) = 1$. $\square$

It is easily seen that we have $C_{d,e}^{(m)} \in \mathrm{CR}^1(d+e,2)$. One may ask whether $C_{d,e}^{(m)}$ is of constant j-rank for $j > 1$. This, in fact, depends on the parameter m as well as the characteristic of k as pointed out in [10, 2.4] and [6, 4.1.6] for modules of the form $C_{2,1}^{(m)}$.
Note that if $\operatorname{char}(k) = p > 0$, $C_{d,e}^{(p)}$ is a subfactor of kE_2 of the form considered in [10, 2.1] with $C_{d,e}^{(p)} \in \mathrm{CJT}(d+e,2)$. However, in case $d+e > p$, the component $\mathcal{C}$ of $\Gamma(d+e,2)$ that contains $C_{d,e}^{(p)}$ is likely to contain $B(d+e,2)$-modules which we can not consider as kE_2-modules.

4 The hereditary case

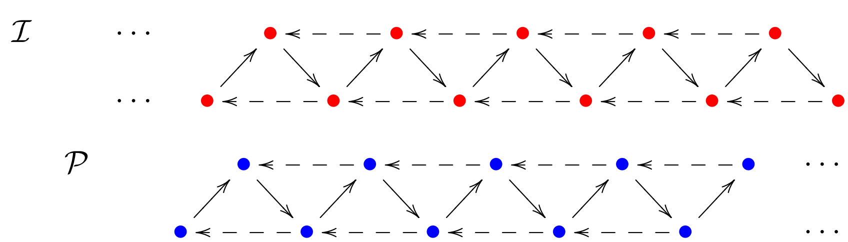
In this section, we confine our investigations to the case $B(2, r)$ where $r \geq 2$. We make use of the fact that $B(2, r)$ is hereditary and furthermore wild, whenever $r > 2$, to show that each $\mathbb{Z}A_\infty$-component of $\Gamma(2, r)$ contains an equal images as well as an equal kernels cone. Furthermore, we show that in case $r > 2$, there is a huge amount of indecomposable modules in $\mathrm{CJT}(2, r)$ that neither satisfy the equal images nor the equal kernels property.

4.1 General facts

The algebra $B(2, r)$ is isomorphic to $\mathcal{K}_r$, the path algebra of the r-Kronecker quiver.

$$\begin{tikzcd} 0 \arrow[r, bend left, "\gamma_1"] \arrow[r, bend right, "\gamma_r"'] \arrow[r, phantom, "\vdots"] & 1 \end{tikzcd}$$

Thus, we are interested in the representation theory of the hereditary algebra $\mathcal{K}_r$, which is wild if $r > 2$ and tame if $r = 2$. Furthermore, note that $\mathfrak{F}_{(2,r)}$ is dense, which has been observed by Heller and Reiner in [23]. Likewise the functors $\mathfrak{F}_\mathcal{X}$ from Proposition 3.10 are dense.
We want to determine the occurrence of modules with the equal images property, the equal kernels property and the constant Jordan type property in the Auslander-Reiten quiver $\Gamma(\mathcal{K}_r) = \Gamma(2, r)$. The shape of the connected components of $\Gamma(\mathcal{K}_r)$ is as follows:
There is exactly one preprojective component $\mathcal{P}$, consisting of the two indecomposable projective modules and their τ^{-1}-shifts and exactly one preinjective component $\mathcal{I}$ consisting of the two indecomposable injective modules and their τ-shifts. Due to Theorem 3.16, we have $\mathcal{I} \subseteq \mathrm{EIP}(2, r)$ and $\mathcal{P} \subseteq \mathrm{EKP}(2, r)$.



Note that there are actually r arrows between any two adjacent vertices in these components. The shape of the regular components depends on the value of r: In case $r = 2$, the regular components are homogeneous tubes, whereas for $r > 2$, they are components of type $\mathbb{Z}A_\infty$. This will be discussed in more detail in the following.

We write $X_\alpha := X_\alpha^1 = \operatorname{coker}\alpha(0)$ for $\alpha \in k^r\backslash 0$. Since X_α neither satisfies the equal images nor the equal kernels property, the indecomposable module X_α is regular. Recall that the modules X_α have distinctive properties due to Proposition 3.13 and Proposition 3.14:

Proposition 4.1. *Let $\alpha \in k^r\backslash 0$. The module X_α is a brick, all proper submodules are projective and there is an equality*

$$Y_\alpha := \tau X_\alpha = \mathrm{D}\, X_\alpha.$$

Moreover, $X_\alpha \cong X_\beta$ if and only if $[\alpha] = [\beta] \in \mathbb{P}^{r-1} = k^r\backslash 0\,/k^\times$.

We denote the component containing the modules X_α and Y_α by $\mathcal{C}_\alpha$. We make a more detailed analysis of the components of type $\mathcal{C}_\alpha$ in Section 4.3 below, where we will make use of the following lemma.

Lemma 4.2. *Let $\alpha, \beta \in k^r\backslash 0$. We have*

(i) $\dim_k \operatorname{Hom}(X_\alpha, Y_\beta) = r - 1$ *if* $[\alpha] = [\beta]$,

(ii) $\dim_k \operatorname{Hom}(X_\alpha, Y_\beta) = r - 2$ *if* $[\alpha] \neq [\beta]$.

Proof. We want to determine a basis of $\operatorname{Hom}(X_\alpha, Y_\beta)$. Note that a map $\varphi \in \operatorname{Hom}(X_\alpha, Y_\beta)$ is uniquely determined by the element $\varphi(\bar{e}_0) \in Y_\beta$ since the module $X_\alpha = P(0)/\alpha(0)(P(1))$ is generated by $\bar{e}_0$. Let $V = \langle \alpha_1\gamma_1 + \cdots + \alpha_r\gamma_r, \beta_1\gamma_1 + \cdots + \beta_r\gamma_r\rangle_k$ and let $\mathcal{B}$ be a basis of $U = \langle \gamma_1, \ldots, \gamma_r\rangle_k/V$, $\mathcal{B}^*$ the dual basis of $U^* = \operatorname{Hom}_k(U, k)$. Since $Y_\beta = \mathrm{D}(X_\beta)$, the module $\mathrm{D}(P(0)/V)$ is a submodule of Y_β, where each $b \in \mathcal{B}^*$ determines a $B(2, r)$-linear map $\varphi : X_\alpha \to Y_\beta$ with $\varphi(\bar{e}_0) = b^*$. These maps are linearly independent and provide a basis of $\operatorname{Hom}(X_\alpha, Y_\beta)$. Since $|\mathcal{B}^*| = \dim_k U$ while $\dim_k U = r - 1$ if $[\alpha] = [\beta]$ and $\dim_k U = r - 2$ if $[\alpha] \neq [\beta]$, the assertion follows. □

Note furthermore that, due to the fact that $\mathcal{K}_r$ is hereditary, we have an isomorphism $\operatorname{Hom}(N, M) \cong \operatorname{Hom}(\tau N, \tau M)$ for any two regular modules $M, N \in \operatorname{mod}\mathcal{K}_r$ and the Auslander-Reiten formula simplifies to

$$\operatorname{Ext}^1(M, N) \cong \big(\operatorname{Hom}(\tau^{-1}N, M)\big)^* \cong (\operatorname{Hom}(N, \tau M))^*.$$

4.2 The tame case: $r = 2$

As was mentioned above, the indecomposable equal images modules for kE_2 of Loewy length at most two have already been classified in [11], namely $\mathrm{EIP}_2(kE_r) = \operatorname{add}\{W_{n,2} \mid n \geq 1\}$. The indecomposable modules in $\mathrm{EIP}(2, 2)$ are the preinjective modules over $\mathcal{K}_2$, which are exactly the modules $W_{n,2}$ and the simple injective module $S(0)$ as pointed out in [16, 4.1.1]. This implies that $\mathrm{EIP}(2, 2)$ is the additive closure of the preinjectives. Dually, the preprojective modules coincide with the indecomposable objects in $\mathrm{EKP}(2, 2)$.

Recall that all regular modules arise as follows (cf. [38, XI.4.6]): For each parameter $\lambda \in k$ and each natural number $m \in \mathbb{N}$, there is a uniquely determined indecomposable module X_m^λ with $\underline{\dim}(X_m^\lambda) = (m, m)$, namely the module which is given by the following representation of the Kronecker quiver:

$$\begin{tikzcd} k^m \arrow[r, bend left, "\mathbb{I}_m"] \arrow[r, bend right, dashed, "\lambda\mathbb{I}_m + J_m"'] & k^m \end{tikzcd}$$

Here $\mathbb{I}_m$ denotes the identity matrix of size $m \times m$ and J_m denotes the nilpotent Jordan block of size $m \times m$. Moreover, we define X_m^∞ to be the representation given by

$$\begin{tikzcd} k^m \arrow[r, bend left, "J_m"] \arrow[r, bend right, dashed, "\mathbb{I}_m"'] & k^m \end{tikzcd}$$

For $\lambda \in k \cup \infty$, we have $\tau_{\mathcal{K}_2}(X_m^\lambda) = X_m^\lambda$ and these modules constitute homogeneous tubes $\mathcal{T}_\lambda$ of $\Gamma(\mathcal{K}_r)$ [38, XI.4.6]:

$$\begin{tikzcd} \mathcal{T}_\lambda & \vdots \arrow[d, bend left] \\ & X_3^\lambda \arrow[u, bend left] \arrow[d, bend left] \arrow[loop left, dashed] \\ & X_2^\lambda \arrow[u, bend left] \arrow[d, bend left] \arrow[loop left, dashed] \\ & X_1^\lambda \arrow[u, bend left] \arrow[loop left, dashed] \end{tikzcd}$$

It is easy to see that the modules in the homogeneous tubes do not satisfy the constant Jordan type property. Note furthermore that there is an isomorphism $X_1^\lambda \cong X_{(-\lambda,1)}$ for all $\lambda \in k$ and an isomorphism $X_1^\infty \cong X_{(1,0)}$. Hence the modules X_α coincide with the modules lying on the mouths of the homogeneous tubes. Since we are in the hereditary situation, we have [2, VIII.2.5]

$$\operatorname{Hom}(\mathcal{I}, \mathcal{T}_\lambda) = 0,$$

implying that all modules in the homogeneous tubes are torsion-free. We can summarize our findings as follows:

Proposition 4.3. *We have the following equalities:*

(i) $\operatorname{EIP}(2,2) = \operatorname{add}\{W_{n,2} \mid n \geq 1\} = \operatorname{add}\mathcal{I}$.

(ii) $\mathrm{EKP}(2,2) = \mathrm{add}\,\{M_{n,2} \mid n \geq 1\} = \mathrm{add}\,\mathcal{P}$.

(iii) $\mathrm{CJT}(2,2) = \mathrm{EIP}(2,2) \oplus \mathrm{EKP}(2,2)$.

(iv) $\mathcal{F}(2,2) = \mathrm{add}\,\mathcal{P} \oplus \bigoplus_{\lambda \in k \cup \infty} \mathrm{add}\,\mathcal{T}_\lambda$, *i.e. the torsion pair* $(\mathrm{EIP}(2,2), \mathcal{F}(2,2))$ *splits.*

The situation in case $r > 2$ is completely different as we will point out in the next section.

4.3 The wild case: $r > 2$

In this section, we assume that $r > 2$. Ringel has proven in [35] that all regular components of $\Gamma(\mathcal{K}_r)$ are of type $\mathbb{Z}A_\infty$. As mentioned above, we have $\mathcal{I} \subseteq \mathrm{EIP}(2,r)$ and $\mathcal{P} \subseteq \mathrm{EKP}(2,r)$ whereas Theorem 3.24 implies that $W^{(r)}_{m,2} \notin \mathcal{I}$ and $M^{(r)}_{m,2} \notin \mathcal{P}$ for $m > 2$. Thus, these modules are examples of regular modules with the equal images property and with the equal kernels property, respectively. Moreover, the torsion pair $(\mathrm{EIP}(2,r), \mathcal{F}(2,r))$ does not split since for $m > 2$, the Auslander-Reiten sequence

$$0 \to \tau M^{(r)}_{m,2} \to E \to M^{(r)}_{m,2} \to 0$$

ending in $M^{(r)}_{m,2}$ satisfies the conditions of Proposition 1.12 and does not split.

In order to show the existence of equal images as well as equal kernels modules in every regular component of $\Gamma(\mathcal{K}_r)$, we record the following dual version of a lemma by Kerner:

Lemma 4.4 (Kerner [26], 4.6)**.** *If X, Y are regular modules over a wild hereditary algebra, there exists an integer N with* $\mathrm{Hom}(Z, \tau^{-m}X) = 0$ *for all $m \geq N$ and all regulars Z with* $\dim_k Z \leq \dim_k Y$.

Note that our next result also follows from Theorem 3.16 in combination with [1, Theorem (B)], a general result concerning non-splitting torsion pairs for wild hereditary algebras.

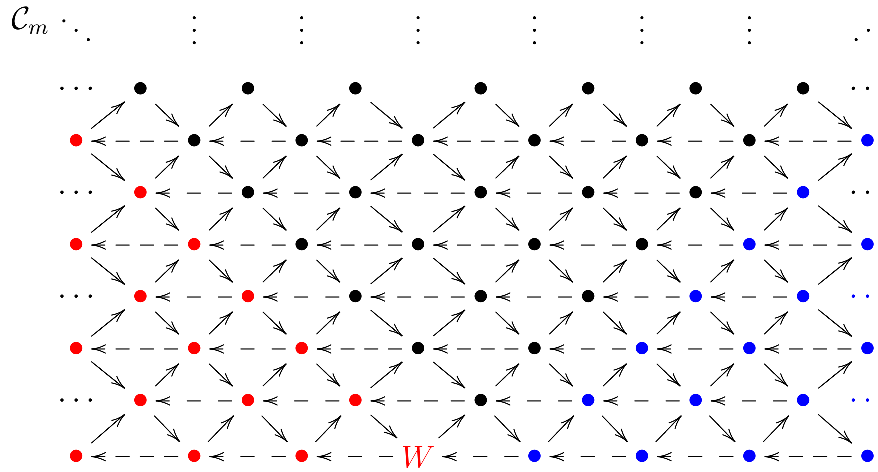
Theorem 4.5. *Let $\mathcal{C}$ be a regular component of $\Gamma(\mathcal{K}_r)$. Then $\mathcal{C}$ contains two uniquely determined quasi-simple modules $W_\mathcal{C}$ and $M_\mathcal{C}$ such that*

$$(\to W_\mathcal{C}) = \mathcal{C} \cap \mathrm{EIP}(2,r) \text{ and } (M_\mathcal{C} \to) = \mathcal{C} \cap \mathrm{EKP}(2,r).$$

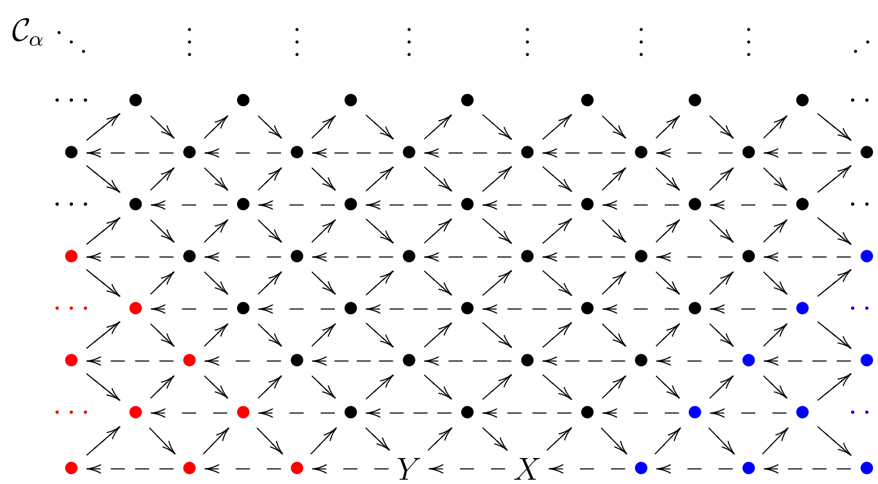
Proof. Let $\mathcal{C}$ be a regular component, X be in $\mathcal{C}$. Let $\alpha \in k^r\backslash 0$ and consider the regular module X_α. Since we have $\dim_k X_\alpha = \dim_k X_\beta$ for all $\beta \in k^r\backslash 0$ and $\mathcal{K}_r$ is wild, we can apply Lemma 4.4 with $Y = X_\alpha$ for some α and Z running through all X_β, $\beta \in k^r\backslash 0$.
This implies that there exists an N such that $\mathrm{Hom}_{\mathcal{K}_r}(X_\alpha, \tau^{-m}X) = 0$ for all $m \geq N$ and all $\alpha \in k^r\backslash 0$. In view of Theorem 3.12 we thus have $\tau^{-m}X \in \mathrm{EKP}(2,r)$ for all $m \geq N$. Dually, $\mathrm{EIP}(2,r) \cap \mathcal{C} \neq \emptyset$. Now apply Proposition 3.20. $\square$

Examples

1. Let $\mathcal{C}_m$ be the component containing the module $W_{m,2}^{(r)}$ for $m > 2$. By Theorem 3.24 we have $\tau^{-1}W_{m,2}^{(r)} \in \mathrm{EKP}(2,r)$ and thus $W_{m,2}^{(r)} = W_{\mathcal{C}_m}$ and $\tau^{-1}W_{m,2}^{(r)} = M_{\mathcal{C}_m}$. Hence $\mathcal{W}(\mathcal{C}_m) = 0$ and the distribution of equal images and equal kernels modules is as follows, where $W = W_{m,2}^{(r)}$:



2. Consider $\mathcal{C}_\alpha$, the component containing X_α. Due to Proposition 4.1, the module X_α is quasi-simple since all proper submodules are projective. Moreover, there is an isomorphism $\tau X_\alpha \cong \mathrm{D}\, X_\alpha = Y_\alpha$. In view of Proposition 3.34, we have $\tau^2 X_\alpha \in \mathrm{EIP}(2,r)$ and dually $\tau^{-2}Y_\alpha \in \mathrm{EKP}(2,r)$. Hence $\mathcal{W}(C_\alpha) = 2$ and the components can be visualized as follows, where $X = X_\alpha$ and $Y = Y_\alpha$:

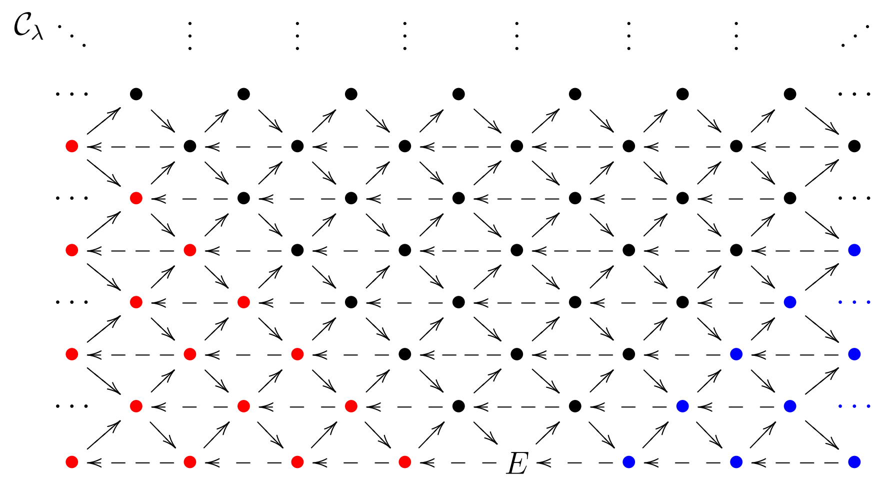


3. Let $\mathcal{C}_\lambda$ be the component containing the brick $E^{(\lambda)}$, $\lambda \in k^r \backslash 0$, with dimension vector $(1,1)$ on which γ_i acts via multiplication with λ_i. The module $E^{(\lambda)}$ is self-dual, i.e. we

have $\mathrm{D}\,E^{(\lambda)} \cong E^{(\lambda)}$. While $E^{(\lambda)} \notin \mathrm{EKP}(2,r)$ and $E^{(\lambda)} \notin \mathrm{EIP}(2,r)$, we have

$$\mathrm{Hom}_{\mathcal{K}_r}(X_\alpha, \tau^{-1}E^{(\lambda)}) \cong \mathrm{Hom}_{\mathcal{K}_r}(\tau X_\alpha, E^{(\lambda)}) \cong \mathrm{Hom}_{\mathcal{K}_r}(Y_\alpha, E^{(\lambda)}) = 0$$

since proper factor modules of Y_α are in $\mathrm{add}\, I(0)$. This implies $\tau^{-1}E^{(\lambda)} \in \mathrm{EKP}(2,r)$ and, dually, we have $\tau E^{(\lambda)} \in \mathrm{EIP}(2,r)$. Hence, we have $\mathcal{W}(\mathcal{C}_\lambda) = 1$ and these components can be visualized as follows, where $E = E^{(\lambda)}$:



These examples show that $\mathcal{W}(\mathcal{C})$ indeed varies while running through the different regular components.

In [25], Kerner has defined an invariant for regular components of the Auslander-Reiten quiver of a wild hereditary algebra A as follows:
Let $\mathcal{C}$ be a regular component of A and X some quasi-simple module in $\mathcal{C}$. The **quasi-rank** of $\mathcal{C}$ is defined via

$$\mathrm{rk}\,\mathcal{C} = \min\left\{m \in \mathbb{Z} |\, \mathrm{rad}(X, \tau^l X) \neq 0 \ \forall l \geq m\right\},$$

where for two indecomposable modules $X, Y \in \mathrm{mod}\,\mathcal{K}_r$, $\mathrm{rad}(X,Y)$ is the vector space of all non-isomorphisms from X to Y (cf. [2, A.3, 3.5]). Hence for $l \neq 0$ and X regular we have $\mathrm{rad}(X, \tau^l X) = \mathrm{Hom}(X, \tau^l X)$.

Ringel has shown in [34] that all regular modules $M \in \mathrm{mod}\,\mathcal{K}_r$ have self-extensions, i.e. $\mathrm{Ext}^1(M,M) \neq 0$. Together with [25, 1.1] this yields that all regular bricks in $\mathrm{mod}\,\mathcal{K}_r$ are quasi-simple. Due to the fact that $\mathrm{Hom}(M,M) \cong \mathrm{Hom}(\tau M, \tau M)$ for any regular $\mathcal{K}_r$-module M, a regular component contains a brick if and only if all quasi-simples are bricks.

Furthermore, for a regular component $\mathcal{C}$ of $\Gamma(\mathcal{K}_r)$, the quasi-rank $\mathrm{rk}\,\mathcal{C}$ is bounded above by 1 and $\mathcal{C}$ contains a brick if and only if $\mathrm{rk}\,\mathcal{C} = 1$ [25, 1.5, 1.7]. The components $\mathcal{C}_m$, $\mathcal{C}_\alpha$ and $\mathcal{C}_\lambda$ show that components containing bricks can, however, be distinguished via the invariant $\mathcal{W}$.

Moreover, we have the following:

Proposition 4.6.

(i) Let $\mathcal{C}$ be a regular component of $\Gamma(\mathcal{K}_r)$. If $\mathcal{C}$ does not possess a brick, then we have $|\operatorname{rk}\mathcal{C}| \leq \mathcal{W}(\mathcal{C})$.

(ii) Let $n \in \mathbb{N}$. Then there exists a regular component $\mathcal{C}$ of $\mathcal{K}_r$ such that $\mathcal{W}(\mathcal{C}) > n$.

Proof. (i): Choose the quasi-simple module $W_{\mathcal{C}}$ in $\mathcal{C}$ given by Theorem 4.5. The module $\tau^{-\mathcal{W}(\mathcal{C})-1}W_{\mathcal{C}} = M_{\mathcal{C}}$ satisfies the equal kernels property and hence by Theorem 3.16 $\operatorname{Hom}(W_{\mathcal{C}}, \tau^{-\mathcal{W}(\mathcal{C})-1}W_{\mathcal{C}}) = 0$, which implies $\operatorname{rk}\mathcal{C} > -\mathcal{W}(\mathcal{C}) - 1$. Since $\mathcal{C}$ does not possess a brick and hence $\operatorname{rk}\mathcal{C} \leq 0$, we have $|\operatorname{rk}\mathcal{C}| \leq \mathcal{W}(\mathcal{C})$.
(ii): In [28, 3.1] it is proven that

$$\inf \{\operatorname{rk}(\mathcal{C}) |\, \mathcal{C} \in \Omega(\mathcal{K}_r)\} = -\infty$$

where $\Omega(\mathcal{K}_r)$ denotes the set of regular components of $\operatorname{mod}\mathcal{K}_r$. Since $\operatorname{rk}\mathcal{C} = 1$ iff $\mathcal{C}$ contains a brick, we can conclude (ii) with (i). □

Note that Proposition 4.6 (i) does not hold if $\mathcal{C}$ contains a brick as can be seen from the example $\mathcal{C}_m$, where we have $\operatorname{rk}\mathcal{C}_m = 1 > 0 = \mathcal{W}(\mathcal{C}_m)$. Moreover, in Proposition 4.10 below, we show that $\mathcal{W}(\mathcal{C})$ is indeed bounded if $\mathcal{C}$ contains a brick.

Let us now consider the category $\operatorname{CJT}(2,r) = \operatorname{CR}^1(2,r)$. We are interested in indecomposable modules that neither satisfy the equal images nor the equal kernels property but satisfy the constant Jordan type property. Recall that due to Proposition 4.3, no such modules exist in case $r = 2$. For $r > 2$, however, the components that are determined by generalized W-modules entirely consist of modules with the constant Jordan type property.

Examples

1. Consider the component $\mathcal{C}_m$ containing $W_{m,2}^{(r)}$ for $m > 2$. We have $\mathcal{W}(\mathcal{C}_m) = 0$ and hence Proposition 3.27 implies that all modules in $\mathcal{C}_m$ have constant Jordan type.

2. Consider the component $\mathcal{C}_\alpha$ containing X_α. We claim that there are no constant rank modules in $\mathcal{C}_\alpha$ apart from the equal images and equal kernels modules. In view of Proposition 3.28, it suffices to show that $[2]X_\alpha$ does not have constant rank. Consider the Auslander-Reiten sequence

$$0 \to Y_\alpha \to [2]X_\alpha \to X_\alpha \to 0.$$

For $\beta \in k^r \backslash 0$, we have an exact sequence

$$0 \to \operatorname{Hom}(X_\beta, Y_\alpha) \to \operatorname{Hom}(X_\beta, [2]X_\alpha) \to \operatorname{Hom}(X_\beta, X_\alpha),$$

where for $[\beta] \neq [\alpha]$ we have $\operatorname{Hom}(X_\beta, X_\alpha) = 0$ and hence due to Lemma 4.2

$$r - 2 = \dim_k \operatorname{Hom}(X_\beta, Y_\alpha) = \dim_k \operatorname{Hom}(X_\beta, [2]X_\alpha).$$

In case $[\beta] = [\alpha]$, we have

$$r - 1 = \dim_k \operatorname{Hom}(X_\beta, Y_\alpha) \leq \dim_k \operatorname{Hom}(X_\beta, [2]X_\alpha).$$

Hence $[2]X_\alpha$ does not have constant rank.

3. Consider the component $\mathcal{C}_\lambda$ containing the module $E^{(\lambda)}$. Since $E^{(\lambda)}$ obviously does not have constant rank and $\mathcal{W}(\mathcal{C}_\lambda) = 1$, Corollary 3.29 implies that there are no modules of constant rank in $\mathcal{C}_\lambda$ apart from the equal kernels and equal images modules.

The examples show that the inclusion $\operatorname{ind}\operatorname{EIP}(2,r) \cup \operatorname{ind}\operatorname{EKP}(2,r) \subseteq \operatorname{ind}\operatorname{CJT}(2,r)$ of indecomposable objects is proper if and only if $r > 2$. Since $\mathfrak{F}_{(2,r)}$ is dense, this directly implies the same result for the categories $\operatorname{EIP}_2(kE_r)$, $\operatorname{EKP}_2(kE_r)$ and $\operatorname{CJT}_2(kE_r)$.

4.4 Refinements of the module categories

For each $\mathbb{Z}A_\infty$-component $\mathcal{C}$ of $\Gamma(\mathcal{K}_r)$, $r > 2$, we have defined an invariant $\mathcal{W}(\mathcal{C})$ counting the number of quasi-simple modules which neither satisfy the equal images nor the equal kernels property. Our current concern is to obtain more information on the modules in the area between the equal images and the equal kernels cone. Due to Proposition 3.27, we already know that a regular component $\mathcal{C}$ entirely consists of modules with the constant Jordan type property provided $\mathcal{W}(\mathcal{C}) = 0$.

Definition 4.7. *Let $M \in \operatorname{mod}\mathcal{K}_r$. We say that M is*

(i) **locally injective** *if there exists $\beta \in k^r\backslash 0$, such that* $\operatorname{Hom}(X_\beta, M) = 0$.

(ii) **locally surjective** *if there exists $\beta \in k^r\backslash 0$, such that* $\operatorname{Ext}^1(X_\beta, M) = 0$.

(iii) **locally bijective** *if M is both locally injective and locally surjective.*

(iv) α**-trivial** *if* $\dim_k \operatorname{Hom}(X_\alpha, M) = \dim_k M_0$ *for $\alpha \in k^r\backslash 0$.*

Note that the modules X_m^λ in the homogeneous tubes $\mathcal{T}_\lambda$ of $\Gamma(\mathcal{K}_2)$ are locally surjective and since $\underline{\dim} X_m^\lambda = (m, m)$, thus locally bijective. Furthermore the modules X_1^λ with $\lambda \in k$ are $(-\lambda, 1)$-trivial and X_1^∞ is $(1, 0)$-trivial. Let us consider $\mathcal{K}_r$-modules with these properties, where $r > 2$.

Proposition 4.8. *Let $r > 2$ and $M \in \operatorname{mod}\mathcal{K}_r$ be indecomposable and locally injective. Then $\tau^{-1}M$ satisfies the equal kernels property. Dually, if M is locally surjective, then τM satisfies the equal images property.*

Proof. Let $\alpha \in k^r\backslash 0$ such that $\operatorname{Hom}(X_\alpha, M) = 0$. Assume there exists $\beta \in k^r\backslash 0$ such that $\operatorname{Hom}(X_\beta, \tau^{-1}M) \neq 0$. Since $\operatorname{Hom}(Y_\beta, M) \cong \operatorname{Hom}(\tau^{-1}Y_\beta, \tau^{-1}M) \cong \operatorname{Hom}(X_\beta, \tau^{-1}M)$, there exists a non-trivial map $\varphi : Y_\beta \to M$. According to Proposition 4.1, proper factor modules of Y_β are injective and since M is indecomposable non-injective, φ must be injective. Since $r > 2$, there exists a non-trivial map

$$\psi : X_\alpha \to Y_\beta$$

in view of Lemma 4.2. The composite $\varphi \circ \psi : X_\alpha \to M$ is non-trivial, which is a contradiction. □

In view of the above, the condition $r > 2$ is necessary, since for $r = 2$ all modules in the homogeneous tubes are counterexamples.
We can generalize the observations we have made about the locally bijective module $E^{(\lambda)}$ subsequent to Theorem 4.5.

Corollary 4.9. *Let $r > 2$, $\mathcal{C}$ be a regular component of $\Gamma(\mathcal{K}_r)$ containing a locally bijective module M. Then M is quasi-simple, $\mathcal{W}(\mathcal{C}) = 1$, and $\mathcal{C}$ contains no other locally bijective module.*

Proof. Proposition 4.8 yields that τM satisfies the equal images property and $\tau^{-1}M$ satisfies the equal kernels property. Hence we have $\mathcal{W}(\mathcal{C}) \leq 1$. Since M is locally injective, we have $\operatorname{Hom}(X_\alpha, M) = 0$ for some $\alpha \in k^r\backslash 0$ but, on the other hand, M is locally surjective and hence not in $\operatorname{EKP}(2, r)$. This implies that M does not have constant rank and in view of Proposition 3.27, we obtain $\mathcal{W}(\mathcal{C}) \neq 0$. Hence $\mathcal{W}(\mathcal{C}) = 1$ and $M \in \mathcal{C}$ is the unique quasi-simple module that does not belong to $\operatorname{EIP}(2, r) \cup \operatorname{EKP}(2, r)$. □

Proposition 4.10. *Let $r > 2$ and let $\mathcal{C}$ be a regular component of $\Gamma(\mathcal{K}_r)$. If $\mathcal{C}$ contains a brick, then $\mathcal{W}(\mathcal{C}) \leq 2$.*

Proof. Let $M \in \mathcal{C}$ be a brick. Recall from Section 4.3 that this implies that M is quasi-simple and that all quasi-simple modules in $\mathcal{C}$ are bricks. The following holds:

(*) If M is not locally injective, then τM satisfies the equal images property.

If M is not locally injective, then for all $\alpha \in k^r\backslash 0$, we have $\operatorname{Hom}(X_\alpha, M) \neq 0$. Assume that $\tau M \notin \operatorname{EIP}(2, r)$, i.e. there exists $\beta \in k^r\backslash 0$ such that $\operatorname{Hom}(M, X_\beta) \cong \operatorname{Ext}^1(X_\beta, \tau M) \neq 0$. Then there exists $\psi \in \operatorname{Hom}(M, X_\beta)\backslash 0$. By our assumption we also have a non-trivial map $\varphi\colon X_\beta \to M$. Since proper submodules of X_β are projective due to Proposition 4.1, ψ is surjective. Hence $\varphi \circ \psi : M \to M$ is non-trivial and not an isomorphism, since M is not isomorphic to the locally injective module X_β. This is a contradiction, since M is a brick. This proves $(*)$. Dually, the module $\tau^{-1}M$ satisfies the equal kernels property if M is not locally surjective.

Let us now consider the module $M := \tau^{-1}W_{\mathcal{C}}$. If M is not locally surjective, then $\tau^{-1}M$ satisfies the equal kernels property, whence $\mathcal{W}(\mathcal{C}) \leq 1$. Thus assume that M is locally surjective. Since M does not satisfy the equal images property, $\tau^{-1}M$ is locally injective with (*). According to Proposition 4.8, $\tau^{-2}M$ satisfies the equal kernels property, whence $\mathcal{W}(\mathcal{C}) \leq 2$.

□

We can make more precise statements about the shape of the connected components of $\Gamma(\mathcal{K}_r)$ containing a so-called elementary module, which is a special type of brick.

Definition 4.11. *Let A be a representation-infinite, hereditary algebra. A regular A-module $E \neq 0$ is called* **elementary** *if there exists no short exact sequence*

$$0 \to N \to E \to M \to 0$$

with N, M being non-zero regular A-modules.

Elementary modules satisfy the property that all modules in their τ-orbit are elementary and, moreover, elementary modules are bricks [29, 1.1, 1.4]. This implies that if a component contains an elementary module, then all its quasi-simple modules are elementary.

Proposition 4.12. *If $\mathcal{C}$ contains an elementary module and $\mathcal{W}(\mathcal{C}) = 2$, then $\mathcal{C} = \mathcal{C}_\alpha$ for some $\alpha \in k^r\backslash 0$.*

Proof. The property $\mathcal{W}(\mathcal{C}) = 2$ implies that $M = \tau M_{\mathcal{C}} \notin \mathrm{EIP}(2, r)$ and hence there exists $\alpha \in k^r\backslash 0$ such that there is a non-trivial and hence surjective map $\varphi : M \to X_\alpha$. Due to the fact that M is elementary and X_α is regular, $P = \ker \varphi$ is preprojective [29, 1.3]. Consider the short exact sequence

$$0 \to P \to M \to X_\alpha \to 0.$$

For $\beta \in k^r\backslash 0$ such that $[\alpha] \neq [\beta]$ there results an exact sequence

$$0 \to \mathrm{Hom}(X_\beta, P) \to \mathrm{Hom}(X_\beta, M) \to \mathrm{Hom}(X_\beta, X_\alpha).$$

Since P is preprojective and hence in $\mathrm{EKP}(2, r)$, we have $\mathrm{Hom}(X_\beta, P) = 0$ and since $[\beta] \neq [\alpha]$, we have $\mathrm{Hom}(X_\beta, X_\alpha) = 0$. Thus we have $\mathrm{Hom}(X_\beta, M) = 0$. Since $M \notin \mathrm{EKP}(2, r)$, this forces $\mathrm{Hom}(X_\alpha, M) \neq 0$. Let $\psi \in \mathrm{Hom}(X_\alpha, M)\backslash 0$. The composite $\psi \circ \varphi$ is non-trivial and since M is a brick, $\psi \circ \varphi$ is an isomorphism. Hence φ is injective and thus an isomorphism. □

Proposition 4.13. *Let M be a regular α-trivial module. Then M is quasi-simple and either $\mathcal{W}(\mathcal{C}) = 1$ or M is isomorphic to X_α or Y_α and $\mathcal{W}(\mathcal{C}) = 2$.*

Proof. Assume that τM does not satisfy the equal images property, i.e. $\mathrm{Ext}^1(X_\beta, \tau M) \neq 0$ for some $\beta \in k^r\backslash 0$. Then there exists a non-trivial and hence surjective morphism $\varphi : M \to X_\beta$. Since $\dim_k \mathrm{Hom}(X_\alpha, M) = \dim_k M_0$ while $\dim_k (X_\alpha)_0 = 1$, this implies the existence of a morphism $\psi : X_\alpha \to M$ such that $\varphi \circ \psi : X_\alpha \to X_\beta$ is non-trivial. This implies $[\alpha] = [\beta]$.

Since X_α is a brick, $\varphi \circ \psi$ is a multiple of the identity on X_α, whence φ is a split epimorphism and $X_\beta \cong X_\alpha$ is a direct summand of M. Since M is indecomposable, we have an isomorphism $M \cong X_\alpha$. Likewise we can show that if $\tau^{-1}M$ does not satisfy the equal kernels property, then we have an isomorphism $M \cong Y_\alpha$. $\square$

Let $M \in \operatorname{mod} \mathcal{K}_r$ and consider the maps $\alpha_M : M \to M$ as defined in Section 3.3. The function $\mathbb{P}^{r-1} \to \mathbb{N}_0, [\alpha] \to \operatorname{rk}(\alpha_M|_{M_0})$ is lower semicontinuous while, dually, the function

$$\varphi_M : \mathbb{P}^{r-1} \to \mathbb{N}_0,\ [\alpha] \mapsto \dim_k \operatorname{Hom}(X_\alpha, M)$$

with $\dim_k \operatorname{Hom}(X_\alpha, M) = \dim_k \ker(\alpha_M|_{M_0})$ is **upper semicontinuous**, i.e. for all $y \in \mathbb{N}_0$, the set $\{[\alpha] \in \mathbb{P}^{r-1} \mid \varphi_M([\alpha]) < y\} \subseteq \mathbb{P}^{r-1}$ is open.

Remark 4.14. Let $M \in \operatorname{mod} \mathcal{K}_r$ with $m_0 = \dim_k M_0$ and $m_1 = \dim_k M_1$. Then

(i) $M \in \operatorname{CJT}(2, r)$ iff φ_M is constant,

(ii) $M \in \operatorname{EKP}(2, r)$ iff $\varphi_M \equiv 0$,

(iii) $M \in \operatorname{EIP}(2, r)$ iff $\varphi_M \equiv m_0 - m_1$.

Moreover, observe the following:

Remark 4.15. Let $M \in \operatorname{mod} \mathcal{K}_r$ with $m_0 = \dim_k M_0$ and $m_1 = \dim_k M_1$. Then M is

(i) locally injective if $\min \varphi_M = 0$.

(ii) locally surjective if $\min \varphi_M = m_0 - m_1$.

(iii) α-trivial for some $\alpha \in k^r \backslash 0$ if $\max \varphi_M = m_0$.

Definition 4.16. *Let $M \in \operatorname{mod} K_r$. We define*

$$\operatorname{MaxRk}(M) := \big\{[\alpha] \in \mathbb{P}^{r-1} \mid \min \varphi_M = \varphi_M([\alpha])\big\} = \varphi_M^{-1}(\min \varphi_M),$$

consisting of the points $[\alpha] \in \mathbb{P}^{r-1}$ with $\operatorname{rk}(\alpha_M|_{M_0})$ maximal.

Remark 4.17. Since φ_M is upper semicontinuous, the subset $\operatorname{MaxRk}(M) \subseteq \mathbb{P}^{r-1}$ is non-empty and open and hence dense in the irreducible space $\mathbb{P}^{r-1}$.

For obvious reasons, we have $\operatorname{MaxRk}(M) = \mathbb{P}^{r-1}$ if and only if M has constant Jordan type, i.e. constant rank. The following can be seen as generalizations of Lemma 3.25 and Lemma 3.26 above

Lemma 4.18. *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be exact.*

(i) If $M_1 \in \operatorname{EIP}(2, r)$, then $\operatorname{MaxRk}(M_2) = \operatorname{MaxRk}(M_3)$.

(ii) If $M_3 \in \operatorname{EKP}(2, r)$, then $\operatorname{MaxRk}(M_1) = \operatorname{MaxRk}(M_2)$.

Proof. We prove (i), (ii) follows dually. Let $M_1 \in \mathrm{EIP}(2,r)$ and $\alpha \in k^r \backslash 0$. Then we have an exact sequence

$$0 \to \mathrm{Hom}(X_\alpha, M_1) \to \mathrm{Hom}(X_\alpha, M_2) \to \mathrm{Hom}(X_\alpha, M_3) \to \mathrm{Ext}^1(X_\alpha, M_1).$$

Since $M_1 \in \mathrm{EIP}(2,r)$ we have $\mathrm{Ext}^1(X_\alpha, M_1) = 0$ and hence $\varphi_{M_2}([\alpha]) = \varphi_{M_1}([\alpha]) + \varphi_{M_3}([\alpha])$. Thus $\mathrm{MaxRk}(M_1) = \mathbb{P}^{r-1}$ implies $\mathrm{MaxRk}(M_3) = \mathrm{MaxRk}(M_2)$. $\square$

Lemma 4.19. *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an Auslander-Reiten sequence. Then*

$$\mathrm{MaxRk}(M_2) = \mathrm{MaxRk}(M_1) \cap \mathrm{MaxRk}(M_3).$$

Proof. Assume first of all that M_3 is not isomorphic to a module of the form X_α. Since the sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ is almost split, we have an exact sequence

$$0 \to \mathrm{Hom}(X_\alpha, M_1) \to \mathrm{Hom}(X_\alpha, M_2) \to \mathrm{Hom}(X_\alpha, M_3) \to 0$$

and hence $\varphi_{M_2}([\alpha]) = \varphi_{M_1}([\alpha]) + \varphi_{M_3}([\alpha])$ for all $\alpha \in k^r \backslash 0$. Since $\mathbb{P}^{r-1}$ is irreducible and the subsets $\mathrm{MaxRk}(M_1), \mathrm{MaxRk}(M_3)$ are non-empty and open, we have $\mathrm{MaxRk}(M_1) \cap \mathrm{MaxRk}(M_3) \neq \emptyset$ and hence $\mathrm{MaxRk}(M_1) \cap \mathrm{MaxRk}(M_3) = \mathrm{MaxRk}(M_2)$.
If M_3, however, is isomorphic to X_α, then $M_1 = Y_\alpha$. We have $\mathrm{MaxRk}(M_3) = \mathbb{P}^{r-1} \backslash [\alpha]$ and in view of Lemma 4.2, we have $\mathrm{MaxRk}(M_1) = \mathbb{P}^{r-1} \backslash [\alpha]$. Moreover, due to Lemma 4.2, we have $\varphi_{M_2}([\beta]) = \varphi_{M_1}([\beta]) + \varphi_{M_3}([\beta]) = r-2$ for $[\beta] \neq [\alpha]$ and $r-1 = \varphi_{M_1}([\alpha]) \leq \varphi_{M_2}([\alpha])$. Hence $\mathrm{MaxRk}(M_2) = \mathbb{P}^{r-1} \backslash [\alpha]$. $\square$

The following is a generalization of Proposition 3.28 and Corollary 3.29:

Proposition 4.20. *Let C be a regular component of $\Gamma(\mathcal{K}_r)$, let $W_{\mathcal{C}}$ and $M_{\mathcal{C}}$ as in Theorem 4.5 and $1 \leq k \leq l$.*

(i) $\mathrm{MaxRk}([k]\tau^{-k} W_{\mathcal{C}}) = \mathrm{MaxRk}([l]\tau^{-k} W_{\mathcal{C}})$ *and* $\mathrm{MaxRk}(\tau^k M_{\mathcal{C}}(k)) = \mathrm{MaxRk}(\tau^k M_{\mathcal{C}}(l))$

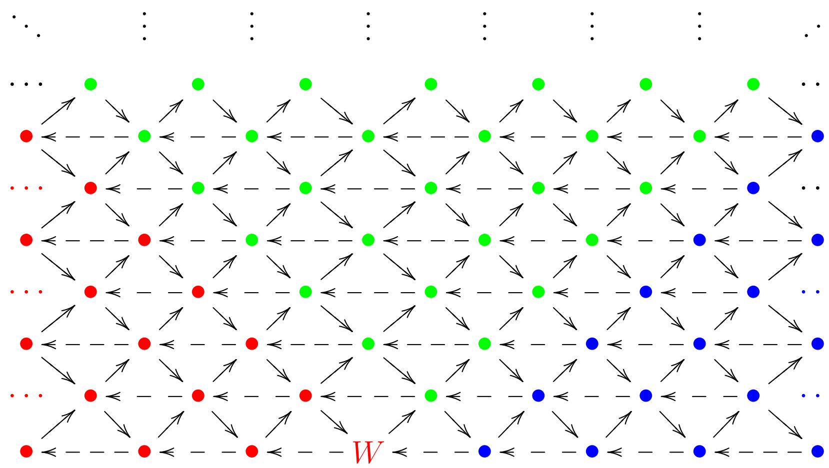
(ii) If $\mathcal{W}(\mathcal{C}) = 1$, *then* $\mathrm{MaxRk}(M) = \mathrm{MaxRk}(N)$ *for all* $M, N \in \mathcal{C} \backslash (\mathrm{EIP}(2,r) \cup \mathrm{EKP}(2,r))$.

Proof. For the proof of (i) consider Lemma 4.18 as a generalization of Lemma 3.25 and use the exact same arguments as in the proof of Proposition 3.28. For (ii) use (i) and the same arguments as in the proof of Corollary 3.29. $\square$

Let us now consider again the examples after Theorem 4.5.

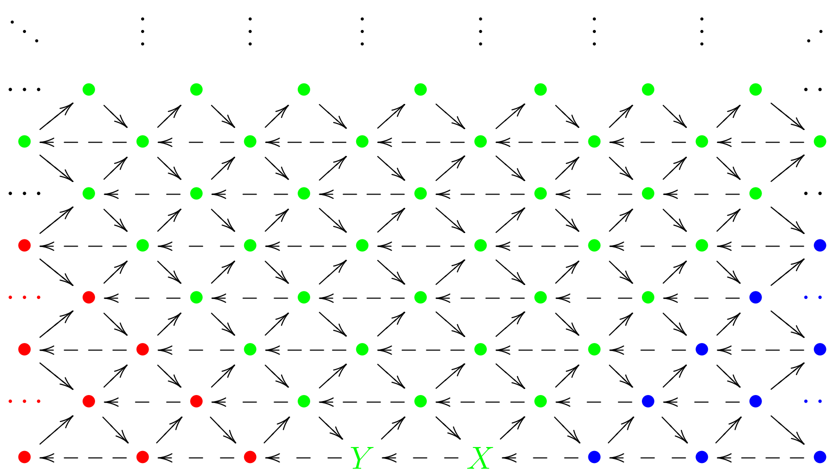
Examples:

1. The component $\mathcal{C}_m$ containing $W = W^{(r)}_{m,2}$ for $m \geq 3$ can be visualized as follows



where $\mathrm{MaxRk}(\textcolor{red}{\bullet}) = \mathrm{MaxRk}(\textcolor{blue}{\bullet}) = \mathrm{MaxRk}(\textcolor{green}{\bullet}) = \mathbb{P}^{r-1}$.

2. Let $\mathcal{C}_\alpha$ be the component containing the quasi-simple bricks $X = X_\alpha$ and $Y = Y_\alpha$ for $\alpha \in k^r \backslash 0$. Then $\mathcal{W}(\mathcal{C}) = 2$ and $\mathcal{C}_\alpha$ can be visualized as follows

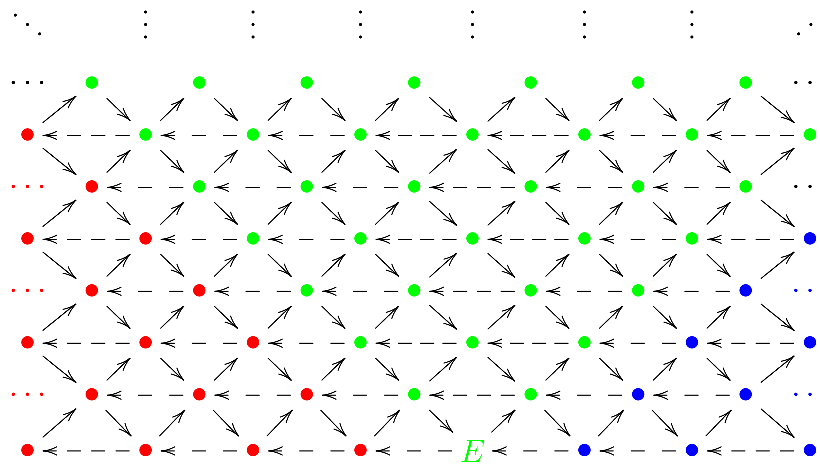


where we have $\mathrm{MaxRk}(\textcolor{red}{\bullet}) = \mathrm{MaxRk}(\textcolor{blue}{\bullet}) = \mathbb{P}^{r-1}$. In the proof of Lemma 4.19, we have shown that

$$\mathrm{MaxRk}([2]X_\alpha) = \mathrm{MaxRk}(X_\alpha) = \mathrm{MaxRk}(Y_\alpha) = \mathbb{P}^{r-1} \backslash [\alpha].$$

Proposition 4.20 (i) now yields that $\mathrm{MaxRk}(\textcolor{green}{\bullet}) = \mathbb{P}^{r-1} \backslash [\alpha]$.

3. Let $\mathcal{C}_\lambda$ be the component containing the brick $E = E^{(\lambda)}$ for $\lambda \in k^r \backslash 0$ with dimension vector $(1,1)$ on which γ_i acts via multiplication with λ_i. Then $\mathcal{C}_\lambda$ can be visualized as follows



where $\mathrm{MaxRk}({\color{red}\bullet}) = \mathbb{P}^{r-1} = \mathrm{MaxRk}({\color{blue}\bullet})$ and $\mathrm{MaxRk}({\color{green}\bullet}) = \mathrm{MaxRk}({\color{green}E^{(\lambda)}})$ due to Proposition 4.20 (ii).
We have $\mathrm{MaxRk}({\color{green}E^{(\lambda)}}) = \{[\alpha] \in \mathbb{P}^{r-1} \mid \sum \lambda_i \alpha_i \neq 0\} = \left\{[\alpha] \in \mathbb{P}^{r-1} \mid \alpha \notin \langle \lambda \rangle^{\perp}\right\}$ due to the fact that $\varphi_{E^{(\lambda)}}([\alpha]) = 0$ if and only if $\sum \lambda_i \alpha_i \neq 0$.

5 Lifting

In this section, we show that generalized W-modules determine $\mathbb{Z}A_\infty$-components in the Auslander-Reiten quiver $\Gamma(n,r)$ of $B(n,r)$ that entirely consist of modules with the constant Jordan type property. We arrive at this result by interpreting the generalized Beilinson algebra as an iterated one-point extensions of the r-Kronecker algebra $\mathcal{K}_r$. This enables us to "lift" our knowledge about $\Gamma(\mathcal{K}_r)$ to a certain extent to $\Gamma(n,r)$. Throughout this section, assume that $n \geq 3$.

5.1 One-point extensions

For a general introduction to the theory of one-point extensions, the reader is referred to [36] or [37, XV.1].

Definition 5.1 (Ringel [36]). *Let A be an algebra, M in* $\operatorname{mod} A$. *The algebra*

$$A[M] = \begin{pmatrix} A & M \\ 0 & k \end{pmatrix}$$

with usual matrix addition and multiplication, i.e.

$$\begin{pmatrix} a & m \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} a' & m' \\ 0 & \lambda' \end{pmatrix} = \begin{pmatrix} aa' & a.m' + \lambda' m \\ 0 & \lambda\lambda' \end{pmatrix}$$

is referred to as the **one-point extension** *of A by M.*

If $A = kQ_A/I$ is a basic algebra, we obtain the quiver $Q_{A[M]}$ of $A[M]$ by adding a source vertex to Q_A.
A module over $A[M]$ is of the form

$$\begin{pmatrix} N \\ V \end{pmatrix}_\varphi,$$

where $N \in \operatorname{mod} A$, $V \in \operatorname{mod} k$ and $\varphi \in \operatorname{Hom}_k(V, \operatorname{Hom}_A(M,N))$. The $A[M]$-module structure is then given via

$$\begin{pmatrix} a & m \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} n \\ v \end{pmatrix} = \begin{pmatrix} a.n + \varphi(v)(m) \\ \lambda v \end{pmatrix}.$$

In case $V = \operatorname{Hom}_A(M,N)$ and $\varphi = \mathrm{id}_{\operatorname{Hom}_A(M,N)}$, we shorten notation and write $\begin{pmatrix} N \\ V \end{pmatrix}_{\mathrm{id}}$.

If $\tilde{X} = \begin{pmatrix} X \\ V \end{pmatrix}_\varphi$, $\tilde{Y} = \begin{pmatrix} Y \\ W \end{pmatrix}_\psi \in \operatorname{mod} A[M]$, then a morphism $\tilde{X} \to \tilde{Y}$ in $\operatorname{mod} A[M]$ corresponds to a pair (f_0, f_1), where $f_0 \in \operatorname{Hom}_A(X,Y)$, $f_1 \in \operatorname{Hom}_k(V,W)$ such that

$$\operatorname{Hom}_A(M, f_0) \circ \varphi = \psi \circ f_1.$$

Since A is a factor algebra of $A[M]$, we have a full exact embedding $\iota_A : \operatorname{mod} A \to \operatorname{mod} A[M]$, sending $N \in \operatorname{mod} A$ to the $A[M]$-module of the form $\begin{pmatrix} N \\ 0 \end{pmatrix}_0$.

On the other hand, we are provided with a functor $\operatorname{res}_A\colon \operatorname{mod} A[M] \to \operatorname{mod} A$ which sends an $A[M]$-module $\begin{pmatrix} N \\ V \end{pmatrix}_\varphi$ to its **restriction** $N \in \operatorname{mod} A$.

There is exactly one simple (injective) module $\tilde{S}$ such that $\operatorname{res}_A(\tilde{S}) = 0$, namely $\begin{pmatrix} 0 \\ k \end{pmatrix}_0$. Moreover, the indecomposable projective $A[M]$-modules are exactly the projective indecomposables of A together with the module

$$P(\tilde{S}) = \begin{pmatrix} M \\ k \end{pmatrix}_{\lambda \mapsto \lambda \operatorname{id}_M},$$

where $\operatorname{rad} P(\tilde{S}) = M$.

In [37, XV.1], a condition is given on when an algebra can be expressed as a one-point extension:

Proposition 5.2. *Let B be a basic algebra and let S be a simple injective B-module with corresponding idempotent e_S. Then B is isomorphic to the one-point extension*

$$\begin{pmatrix} (1-e_S)B(1-e_S) & \operatorname{rad} P(S) \\ 0 & k \end{pmatrix},$$

where $(1-e_S)B(1-e_S) \cong B/Be_SB$.

Here, $N \in \operatorname{mod} B$ corresponds to the $(1-e_S)B(1-e_S)[\operatorname{rad} P(S)]$-module $\begin{pmatrix} (1-e_S)N \\ e_SN \end{pmatrix}_{\psi\circ\theta}$ with $\theta\colon e_SN \to \operatorname{Hom}_B(P(S), N)$ the canonical isomorphism of vector spaces and with $\psi = \operatorname{res}_{(1-e_S)B(1-e_S)}\colon \operatorname{Hom}_B(P(S), N) \to \operatorname{Hom}_{(1-e_S)B(1-e_S)}(\operatorname{rad} P(S), (1-e_S)N)$.

In view of the Auslander-Reiten theory of $A[M]$, the following lemma gives information on how almost split sequences in $\operatorname{mod} A$ "lift" to $\operatorname{mod} A[M]$ [36, 2.5].

Lemma 5.3. *Let A be an algebra, M an A-module. Let furthermore*

$$0 \to \tau N \xrightarrow{f} E \xrightarrow{g} N \to 0$$

be an Auslander-Reiten sequence in $\operatorname{mod} A$. *Then*

$$0 \to \begin{pmatrix} \tau N \\ \operatorname{Hom}_A(M, \tau N) \end{pmatrix}_{\operatorname{id}} \xrightarrow{\begin{pmatrix} f \\ \operatorname{id} \end{pmatrix}} \begin{pmatrix} E \\ \operatorname{Hom}_A(M, \tau N) \end{pmatrix}_{f^*} \xrightarrow{\begin{pmatrix} g \\ 0 \end{pmatrix}} \begin{pmatrix} N \\ 0 \end{pmatrix}_0 \to 0$$

is an Auslander-Reiten sequence in $\operatorname{mod} A[M]$.

These "lifted" sequences are in fact the only Auslander-Reiten sequences that do not split after restriction to A via res_A [36, 2.5].

5.2 Beilinson algebras as iterated one-point extensions

Let us now show that we can regard the algebra $B(n,r)$ as a one-point extension of $B(n-1,r)$. The simple module $S(0) \in \operatorname{mod} B(n,r)$ is injective. Recall that the module $M_{n,n}^{(r)} \in \operatorname{mod} B(n,r)$ is isomorphic to the projective indecomposable module $P(0) \in \operatorname{mod} B(n,r)$ and due to Lemma 3.23 (i), there is an isomorphism $\operatorname{rad} P(0) \cong M_{n,n-1}^{(r)}$.
Note, moreover, that the algebra $B(n-1,r)$ is isomorphic to the algebra $(1-e_0)B(n,r)(1-e_0)$ which yields the following isomorphism of algebras in view of Proposition 5.2:

$$B(n,r) \cong B(n-1,r)[M_{n,n-1}^{(r)}] \cong \mathcal{K}_r[M_{3,2}^{(r)}]\cdots[M_{n,n-1}^{(r)}] \tag{9}$$

For $n=3$, $r=2$, let us visualize this as follows: Extending the path algebra $\mathcal{K}_2$ of the quiver

$$\begin{tikzcd} 1 \arrow[r, bend left, "\gamma_1"] \arrow[r, bend right, dashed, "\gamma_2"'] & 2 \end{tikzcd}$$

by the module $M_{3,2}^{(2)}$

$$\begin{tikzcd} & \bullet^{x_1} \arrow[dl] \arrow[dr, dashed] & & \bullet^{x_2} \arrow[dl] \arrow[dr, dashed] & \\ \bullet & & \bullet & & \bullet \end{tikzcd}$$

yields the path algebra of

$$\begin{tikzcd} 0 \arrow[r, bend left, "x_1"] \arrow[r, bend right, dashed, "x_2"'] & 1 \arrow[r, bend left, "\gamma_1"] \arrow[r, bend right, dashed, "\gamma_2"'] & 2 \end{tikzcd}$$

with relations $\gamma_2.x_1 = \gamma_1.x_2$, which is easily seen to be isomorphic to $B(3,2)$.

From now on, we will identify the algebras $B(n,r)$ and $B(n-1,r)[M_{n,n-1}^{(r)}]$. Note that when writing $\tilde{M} \in \operatorname{mod} B(n,r)$ in the form $\tilde{M} = \begin{pmatrix} M \\ V \end{pmatrix}_{\varphi} \in \operatorname{mod} B(n-1,r)[M_{n,n-1}^{(r)}]$, the dimension vector $\underline{\dim}\tilde{M}$ coincides with the vector $(\dim_k V, \underline{\dim} M)$. Moreover, the functor $\operatorname{res}_{B(n-1,r)}$ is equivalent to the functor Ψ_n^{n-1} from Section 3.8.

We are ultimately interested in the Auslander-Reiten theory of $\Gamma(n,r)$ and in view of (9) and Lemma 5.3, we want to make use of the information we have about $\Gamma(\mathcal{K}_r)$ provided in Chapter 4.

Considering Lemma 5.3, we thus take a closer look at the functor $\operatorname{Hom}_{B(n-1,r)}(M_{n,n-1}^{(r)}, -)$. We want to determine the lifts of the Auslander-Reiten sequences starting in generalized M- and W-modules, respectively, and therefore show the following:

Lemma 5.4. *Let $m \geq n \geq 3$. There is an isomorphism of vector spaces*

$$\operatorname{Hom}_{B(n-1,r)}(M_{n,n-1}^{(r)}, M_{m,n-1}^{(r)}) \cong k[X_1,\ldots,X_r]_{m-n}$$

and an isomorphism

$$\begin{pmatrix} M^{(r)}_{m,n-1} \\ \mathrm{Hom}_{B(n-1,r)}(M^{(r)}_{n,n-1}, M^{(r)}_{m,n-1}) \end{pmatrix}_{\mathrm{id}} \cong M^{(r)}_{m,n}$$

in $\mathrm{mod}\, B(n,r)$.

Proof. In view of the equivalence $\mathcal{D}_{[0,n-2]} \cong \mathrm{mod}\, B(n-1,r)$ given in Section 3.6, there is an isomorphism

$$\mathrm{Hom}_{B(n-1,r)}(M^{(r)}_{n,n-1}, M^{(r)}_{m,n-1}) \cong \mathrm{Hom}^{\mathbb{Z}}_R(M^{(r)}_{n,n-1}[-1], M^{(r)}_{m,n-1}[n-1-m]).$$

Directly from the definition of the shift-functor, we furthermore obtain

$$\mathrm{Hom}^{\mathbb{Z}}_R(M^{(r)}_{n,n-1}[-1], M^{(r)}_{m,n-1}[n-1-m]) = \mathrm{Hom}^{\mathbb{Z}}_R(M^{(r)}_{n,n-1}, M^{(r)}_{m,n-1}[n-m]).$$

Consider the module $M^{(r)}_{m,m-1} \in \mathrm{mod}_{\mathbb{Z}}\, R$. There is an isomorphism

$$M^{(r)}_{n,n-1} \cong (M^{(r)}_{m,m-1})_{<n}$$

and an equality $M^{(r)}_{m,n-1} = (M^{(r)}_{m,m-1})_{\geq m-n+1}$ and thus $M^{(r)}_{m,n-1}[n-m] = (M^{(r)}_{m,m-1}[n-m])_{\geq 1}$ in $\mathrm{mod}_{\mathbb{Z}}\, R$. Hence

$$\mathrm{Hom}^{\mathbb{Z}}_R(M^{(r)}_{n,n-1}, M^{(r)}_{m,n-1}[n-m]) \cong \mathrm{Hom}^{\mathbb{Z}}_R(M^{(r)}_{m,m-1}, M^{(r)}_{m,m-1}[n-m]) = \mathrm{End}_R(M^{(r)}_{m,m-1})_{m-n}.$$

It can be seen from the proof of Proposition 2.13 that the isomorphism of vector spaces

$$k[X_1, \ldots, X_r]_{m-n} \cong \mathrm{End}_R(M^{(r)}_{m,m-1})_{m-n} \cong \mathrm{Hom}_{B(n-1,r)}(M^{(r)}_{n,n-1}, M^{(r)}_{m,n-1})$$

is given by

$$\rho : k[X_1, \ldots, X_r]_{m-n} \to \mathrm{Hom}_{B(n-1,r)}(M^{(r)}_{n,n-1}, M^{(r)}_{m,n-1}),\ f \mapsto \left\{ \begin{array}{ll} \rho_f: & M^{(r)}_{n,n-1} \to M^{(r)}_{m,n-1}, \\ & m + I^{n-1} \mapsto mf + I^{n-1} \end{array} \right. .$$

The $B(n,r)$-module $M^{(r)}_{m,n}$ is generated by $e_0.M^{(r)}_{m,n} = k[X_1, \ldots, X_r]_{m-n}$ and we can identify $M^{(r)}_{m,n} \in \mathrm{mod}\, B(n,r)$ with the $B(n-1,r)[M^{(r)}_{n,n-1}]$-module

$$\begin{pmatrix} M^{(r)}_{m,n-1} \\ k[X_1, \ldots, X_r]_{m-n} \end{pmatrix}_{\rho}.$$

Now ρ induces an isomorphism between the $B(n-1,r)[M^{(r)}_{n,n-1}]$-modules $\begin{pmatrix} M^{(r)}_{m,n-1} \\ k[X_1, \ldots, X_r]_{m-n} \end{pmatrix}_{\rho}$ and $\begin{pmatrix} M^{(r)}_{m,n-1} \\ \mathrm{Hom}_{B(n-1,r)}(M^{(r)}_{n,n-1}, M^{(r)}_{m,n-1}) \end{pmatrix}_{\mathrm{id}}$ given by the pair $(\mathrm{id}_{M^{(r)}_{m,n-1}}, \rho)$. This yields the assertion. $\square$

Moreover, we can show:

Lemma 5.5. *Let $m \geq n \geq 3$. There is an isomorphism of vector spaces*

$$\operatorname{Hom}_{B(n-1,r)}(M^{(r)}_{n,n-1}, W^{(r)}_{m,n-1}) \cong (k[X_1, \ldots, X_r]_m)^*$$

and an isomorphism

$$\begin{pmatrix} W^{(r)}_{m,n-1} \\ \operatorname{Hom}_{B(n-1,r)}(M^{(r)}_{n,n-1}, W^{(r)}_{m,n-1}) \end{pmatrix}_{\mathrm{id}} \cong W^{(r)}_{m+1,n}$$

in $\operatorname{mod} B(n,r)$.

Proof. Due to Lemma 3.23, there is an equality $M^{(r)}_{n,n-1} = \operatorname{rad} P(0)$ in $\operatorname{mod} B(n,r)$ and furthermore we have $\iota_{B(n-1,r)}(W^{(r)}_{m,n-1}) \cong \operatorname{rad} W^{(r)}_{m+1,n}$ in $\operatorname{mod} B(n,r)$. In view of the fact that $\iota_{B(n-1,r)}$ is fully faithful and the fact that all morphisms $\operatorname{rad} P(0) \to W^{(r)}_{m+1,n}$ factor through $\operatorname{rad} W^{(r)}_{m+1,n}$, there is an isomorphism

$$\operatorname{Hom}_{B(n,r)}(\operatorname{rad} P(0), W^{(r)}_{m+1,n}) \cong \operatorname{Hom}_{B(n-1,r)}(M^{(r)}_{n,n-1}, W^{(r)}_{m,n-1}).$$

Now consider the exact sequence of Hom-spaces in $\operatorname{mod} B(n,r)$:

$$\operatorname{Hom}(S(0), W^{(r)}_{m+1,n}) \to \operatorname{Hom}(P(0), W^{(r)}_{m+1,n}) \to \operatorname{Hom}(\operatorname{rad} P(0), W^{(r)}_{m+1,n}) \to \operatorname{Ext}^1(S(0), W^{(r)}_{m+1,n})$$

Since $S(0)$ is injective, we have $\operatorname{Hom}(S(0), W^{(r)}_{m+1,n}) = 0$. Furthermore, we have

$$\operatorname{Ext}^1(S(0), W^{(r)}_{m+1,n}) \cong \operatorname{Ext}^1(\mathrm{D}\, W^{(r)}_{m+1,n}, \mathrm{D}\, S(0)) \cong \operatorname{Ext}^1(M^{(r)}_{m+1,n}, S(n-1)).$$

Due to Lemma 3.23 we have $\operatorname{Ext}^1(M^{(r)}_{m+1,n}, S(n-1)) = 0$ since $n-1 \neq 1$. This yields $\operatorname{Ext}^1_{B(n,r)}(S(0), W^{(r)}_{m+1,n}) = 0$ and the restriction functor $\operatorname{res}_{B(n-1,r)}$ provides an isomorphism

$$\psi\colon \operatorname{Hom}_{B(n,r)}(P(0), W^{(r)}_{m+1,n}) \to \operatorname{Hom}_{B(n-1,r)}(M^{(r)}_{n,n-1}, W^{(r)}_{m,n-1})$$

of vector spaces. Moreover, we have

$$e_0.W^{(r)}_{m+1,n} = (k[X_1, \ldots, X_r]_m)^*$$

and the canonical isomorphism of vector spaces

$$\delta : (k[X_1, \ldots, X_r]_m)^* \to \operatorname{Hom}_{B(n,r)}(P(0), W^{(r)}_{m+1,n}), f^* \mapsto \left\{ \begin{array}{c} \eta_{f^*}\colon P(0) \to W^{(r)}_{m+1,n} \\ e_0 \mapsto f^* \end{array} \right.$$

induces an isomorphism of $B(n-1,r)[M^{(r)}_{n,n-1}]$-modules

$$\begin{pmatrix} W^{(r)}_{m,n-1} \\ (k[X_1, \ldots, X_r]_m)^* \end{pmatrix}_{\psi\delta} \cong \begin{pmatrix} W^{(r)}_{m,n-1} \\ \operatorname{Hom}_{B(n-1,r)}(M^{(r)}_{n,n-1}, W^{(r)}_{m,n-1}) \end{pmatrix}_{\mathrm{id}}$$

given by the pair $(\mathrm{id}_{W^{(r)}_{m,n-1}}, \psi\delta)$. The module on the left-hand side corresponds to the $B(n,r)$-module $W^{(r)}_{m+1,n}$, which yields the assertion. $\square$

In view of Proposition 5.3, we obtain the following:

Proposition 5.6. *Let $m \geq n \geq 3$. We have*

(i) $\iota_{B(n-1,r)}\tau^{-1}_{B(n-1,r)}(M^{(r)}_{m,n-1}) \cong \tau^{-1}_{B(n,r)}M^{(r)}_{m,n}$,

(ii) $\iota_{B(n-1,r)}\tau^{-1}_{B(n-1,r)}(W^{(r)}_{m,n-1}) \cong \tau^{-1}_{B(n,r)}W^{(r)}_{m+1,n}$.

Hence the Auslander-Reiten sequences in $\operatorname{mod} B(n-1,r)$ starting in $M^{(r)}_{m,n-1}$ and $W^{(r)}_{m,n-1}$ lift to Auslander-Reiten sequences in $\operatorname{mod} B(n,r)$ that start in $M^{(r)}_{m,n}$ and $W^{(r)}_{m+1,n}$, respectively.

5.3 Occurrence of generalized W-modules in $\Gamma(n,r)$

We show that generalized W-modules determine $\mathbb{Z}A_\infty$-components in $\Gamma(n,r)$, $n \geq 3$, $r \geq 2$, that entirely consist of modules with the constant Jordan type property.

Let us consider the case $r = 2$. On the level of the Auslander-Reiten quiver $\Gamma(2,2)$ of $\mathcal{K}_2$, we do not have any $\mathbb{Z}A_\infty$-components to start out with and the W-modules correspond to the preinjective $\mathcal{K}_2$-modules.
At this point, let me add that with the use of tilting theory one can show that all regular components of $\Gamma(3,2)$ are of type $\mathbb{Z}A_\infty$ as has been communicated to me by Otto Kerner [27]: There exists a preprojective tilting module T over the path algebra of the extended Kronecker quiver

$$0 \longrightarrow 1 \rightrightarrows 2$$

such that $\operatorname{End}(T)$ is isomorphic to $B(3,2)$ while the regular components of $\Gamma(\operatorname{End}(T))$ are of type $\mathbb{Z}A_\infty$.

We are, however, interested in determining components of $\Gamma(n,r)$ for all values of $n, r \geq 2$, where these methods are not applicable.

Consider the preinjective $\mathcal{K}_2$-modules, i.e. modules of the form $W^{(2)}_{m,2}$ where $m \geq 1$. Proposition 5.6 (ii) specializes to $\tau^{-1}_{B(3,2)}W^{(2)}_{m+3,3} = \iota_{B(2,2)}\tau^{-1}_{\mathcal{K}_2}(W^{(2)}_{m+2,2})$ and hence in view of the Auslander-Reiten sequence

$$\Sigma:\ 0 \to W^{(2)}_{m+2,2} \to W^{(2)}_{m+1,2} \oplus W^{(2)}_{m+1,2} \to W^{(2)}_{m,2} \to 0$$

in $\operatorname{mod} \mathcal{K}_2$, we obtain

$$\tau^{-1}_{B(3,2)}(W^{(2)}_{m+3,3}) = \iota_{B(2,2)}W^{(2)}_{m,2}.$$

Proposition 5.6 (ii) now inductively yields

$$\tau^{-1}_{B(n,2)}(W^{(2)}_{m+n,n}) = \iota_{B(n-1,2)} \cdots \iota_{B(2,2)}(W^{(2)}_{m,2})$$

and dually

$$\tau_{B(n,2)}(M^{(2)}_{m+n,n}) = \mathrm{D}\,\iota_{B(n-1,2)} \cdots \iota_{B(2,2)}(W^{(2)}_{m,2}).$$

Theorem 5.7. *Let $m > n \geq 3$ and let $\mathcal{C}^{(2)}_m$ of $\Gamma(n,2)$ be the component containing $W^{(2)}_{m,n}$, Then $\mathcal{C}^{(2)}_m$ is a regular $\mathbb{Z}A_\infty$-component such that $\mathcal{C}^{(2)}_m \subseteq \mathrm{CJT}(n,2)$ and $\mathcal{W}(\mathcal{C}^{(2)}_m) = 1$.*

Proof. We consider the dual component $\mathcal{D}^{(2)}_m = \mathrm{D}\,\mathcal{C}^{(2)}_m$ containing the module $\mathrm{D}\,W^{(2)}_{m,n} = M^{(2)}_{m,n}$. In view of the duality between $\mathrm{EIP}(n,2)$ and $\mathrm{EKP}(n,2)$ and the fact that the constant Jordan type property is self-dual, it suffices to prove the assertion for $\mathcal{D}^{(2)}_m$.
Note that for the $B(n,2)$-module $\mathrm{D}\,\iota_{B(n-1,2)} \cdots \iota_{B(2,2)}(W^{(2)}_{m-n,2})$, we have

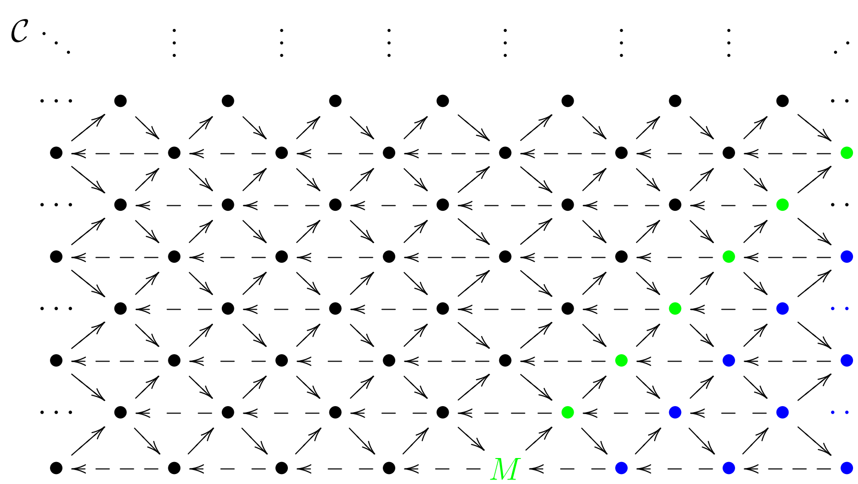
$$\Phi^2_n \left(\mathrm{D}\,\iota_{B(n-1,2)} \cdots \iota_{B(2,2)}(W^{(2)}_{m-n,2})\right) \cong \mathrm{D}\,W^{(2)}_{m-n,2} \cong M^{(2)}_{m-n,2} \in \mathrm{EKP}(2,2)$$

while

$$\Psi^{n-1}_n \left(\mathrm{D}\,\iota_{B(n-1,2)} \cdots \iota_{B(2,2)}(W^{(2)}_{m-n,2})\right) \cong \tilde{S}(0)^{\oplus m-n} \in \mathrm{EIP}(n-1,2).$$

Hence by Proposition 3.37 for $k = 2$, we obtain that $\mathrm{D}\,\iota_{B(n-1,2)} \cdots \iota_{B(2,2)}(W^{(2)}_{m-n,2})$ and thus also $M^{(2)}_{m,n}$ is a quasi-simple module in the $\mathbb{Z}A_\infty$-component $\mathcal{D}^{(2)}_m$ of $\Gamma(n,2)$ with $\mathcal{W}(\mathcal{D}^{(2)}_m) = 1$. Hence $\mathcal{D}^{(2)}_m$ is a component as in Definition 3.21.
Since $\mathrm{D}\,\iota_{B(n-1,2)} \cdots \iota_{B(2,2)}(W^{(2)}_{m-n,2}) \in \mathrm{CJT}(n,2)$ due to the fact that $W^{(2)}_{m-n,2} \in \mathrm{CJT}(2,2)$, while $\mathrm{D}\,\iota_{B(n-1,2)} \cdots \iota_{B(2,2)}(W^{(2)}_{m-n,2}) \notin \mathrm{EIP}(n,2) \cup \mathrm{EKP}(n,2)$, we obtain $\mathcal{D}^{(2)}_m \subseteq \mathrm{CJT}(n,2)$ in view of Corollary 3.29. $\square$

Let us now consider the case $r > 2$, where all regular components of $\Gamma(2,r)$ are of type $\mathbb{Z}A_\infty$. Suppose we have a $\mathbb{Z}A_\infty$-component $\mathcal{C}$ in $\Gamma(n-1,r)$ containing a module with the equal kernels property but which at the same time is not completely contained in the category $\mathrm{EKP}(n-1,r)$. Then by Proposition 3.20, $\mathcal{C}$ contains an equal kernels cone $(M_\mathcal{C} \to)$ consisting of a distinct quasi-simple module $M = M_\mathcal{C} \in \mathrm{EKP}(n,r)$ and all its successors in $\mathcal{C}$.



Proposition 5.8. *Let $\mathcal{C}$ be a component of $\Gamma(n-1,r)$ as above and let*

$$0 \to \tau N \to E \to N \to 0 \tag{10}$$

be an Auslander-Reiten sequence in the subcone $(\tau^{-1}M_{\mathcal{C}} \to)$ of the equal kernels cone $(M_{\mathcal{C}} \to)$. Then

$$0 \to \iota_{B(n-1,r)}(\tau N) \to \iota_{B(n-1,r)}(E) \to \iota_{B(n-1,r)}(N) \to 0$$

is an Auslander-Reiten sequence in $\Gamma(n,r)$.

Proof. In order to determine the lift of (10) to $\operatorname{mod} B(n,r)$, we need to compute

$$\operatorname{Hom}_{B(n-1,r)}(M^{(r)}_{n,n-1}, \tau N)$$

in view of Lemma 5.3. We have $\tau M^{(r)}_{n,n-1} \in \operatorname{EIP}(n-1,r)$ according to Theorem 3.24, whereas $\tau^2 N \in \operatorname{EKP}(n-1,r)$, since $\tau N \in (\tau^{-1}M_{\mathcal{C}} \to)$. Hence the Auslander-Reiten formula yields an isomorphism of vector spaces

$$0 = \overline{\operatorname{Hom}}_{B(n-1,r)}(\tau M^{(r)}_{n,n-1}, \tau^2 N) \cong \underline{\operatorname{Hom}}_{B(n-1,r)}(M^{(r)}_{n,n-1}, \tau N). \tag{11}$$

The module $M^{(r)}_{n,n-1}$ is indecomposable non-projective and generated by $(M^{(r)}_{n,n-1})_0$ while $(P(i))_0 = 0$ for $0 < i \leq n-1$ and $\dim_k(P(0))_0 = 1$. Hence we have $\operatorname{Hom}(M^{(r)}_{n,n-1}, P(i)) = 0$ for all $0 \leq i \leq n-1$. In view of (11), we thus we obtain $\operatorname{Hom}_{B(n-1,r)}(M^{(r)}_{n,n-1}, \tau N) = 0$ and in view of Lemma 5.3, the sequence (10) lifts to the Auslander-Reiten sequence

$$0 \to \begin{pmatrix} \tau N \\ 0 \end{pmatrix}_0 \to \begin{pmatrix} E \\ 0 \end{pmatrix}_0 \to \begin{pmatrix} N \\ 0 \end{pmatrix}_0 \to 0 \tag{12}$$

in $\operatorname{mod} B(n,r)$. □

On the other hand, consider the sequence

$$0 \to M_{\mathcal{C}} \to E \to \tau^{-1}M_{\mathcal{C}} \to 0. \tag{13}$$

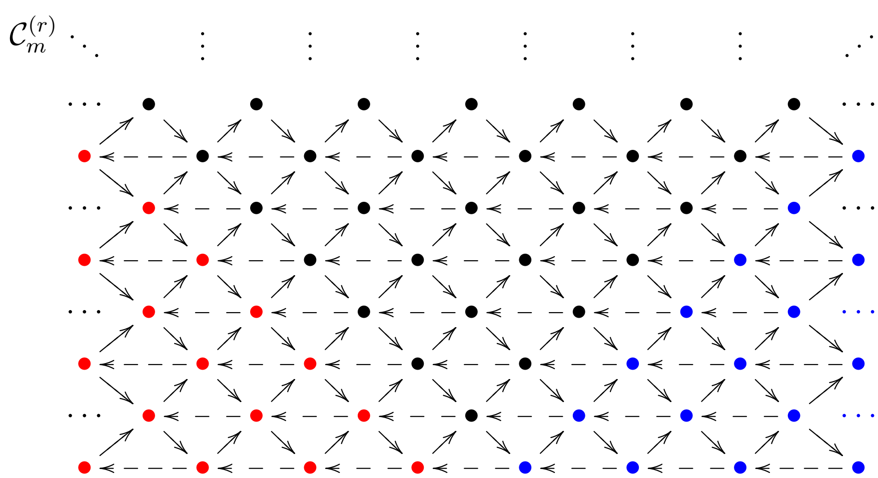
Due to the fact that $\tau M_{\mathcal{C}} \notin \operatorname{EKP}(n-1,r)$, we do not necessarily obtain $\operatorname{Hom}_{B(n-1,r)}(M^{(r)}_{n,n-1}, M_{\mathcal{C}}) = 0$. In particular, Proposition 5.6 implies that the component $\hat{\mathcal{D}}_n$ of $\Gamma(n-1,r)$ that contains the module $M^{(r)}_{n,n-1}$ gives rise to a component $\mathcal{D}^{(r)}_n$ of $\Gamma(n,r)$ containing the projective module $P(0) = M^{(r)}_{n,n} \in \operatorname{mod} B(n,r)$.

Theorem 5.9. *Let $r > 2$ and let $n \geq 2$, $m > n$. The module $W_{m,n}^{(r)}$ belongs to a regular $\mathbb{Z}A_\infty$-component $\mathcal{C}_m^{(r)}$ of $\Gamma(n,r)$ such that $\mathcal{W}(\mathcal{C}_m^{(r)}) = 0$.*

Proof. We consider the dual component $\mathcal{D}_m^{(r)} = \mathrm{D}\,\mathcal{C}_m^{(r)}$ containing the module $\mathrm{D}\,W_{m,n}^{(r)} = M_{m,n}^{(r)}$. In view of the duality between $\mathrm{EIP}(n,r)$ and $\mathrm{EKP}(n,r)$ and the compatibility between τ and D, it suffices to prove the assertion for $\mathcal{D}_m^{(r)}$. As shown in Example 1 after Theorem 4.5, the statement holds for $n = 2$. Now assume that $n \geq 3$ and the statement is true for $n-1$. In view of Proposition 5.6, the regular $\mathbb{Z}A_\infty$-component $\hat{\mathcal{D}}_m$ of $\Gamma(n-1,r)$ containing $M_{\hat{\mathcal{D}}_m} = M_{m,n-1}^{(r)}$ gives rise to a component $\mathcal{D}_m^{(r)}$ of $\Gamma(n,r)$ containing the module $M_{m,n}^{(r)}$, where the cone $(\tau^{-1}M_{m,n}^{(r)} \rightarrow)$ coincides via $\iota_{B(n-1,r)}$ with the cone $(\tau^{-1}M_{m,n-1}^{(r)}) \rightarrow) \subseteq \Gamma(n-1,r)$ due to Proposition 5.8.

Let $\mathcal{D} = \left\{\tau^k X \mid X \in (M_{m,n}^{(r)} \rightarrow),\ k \in \mathbb{Z}\right\} \subseteq \mathcal{D}_m^{(r)}$. Due to the fact that $m > n$, $\mathcal{D}$ does not contain $P(0)$ and hence $\mathcal{D}$ does not contain any projective vertices since all modules in $(\tau^{-1}M_{m,n-1}^{(r)} \rightarrow)$ are regular $B(n-1,r)$-modules. Furthermore $\mathcal{D}$ is τ-stable as well as τ^{-1}-stable and for $X \in \mathcal{D}$, we have $Y \in \mathcal{D}$ if there is an irreducible map $X \rightarrow Y$ or $Y \rightarrow X$. Since $\mathcal{D}_m^{(r)}$ is connected we have $\mathcal{D}_m^{(r)} = \mathcal{D}$. Due to the fact that $\tau^k M_{m,n}^{(r)} \in \mathrm{EIP}(n,r)$ if and only if $k > 0$, $\mathcal{D}_m$ is non-periodic and hence $\mathcal{D}_m^{(r)}$ is a $\mathbb{Z}A_\infty$-component. According to Theorem 3.24, we have $\mathcal{W}(\mathcal{D}_m^{(r)}) = 0$. $\square$

The distribution of equal images and equal kernels modules is as follows:



5.4 More $\mathbb{Z}A_\infty$-components

We show that the modules X_m^λ in the homogeneous tubes $\mathcal{T}_\lambda$ of $\Gamma(\mathcal{K}_2)$ determine $\mathbb{Z}A_\infty$-components $\mathcal{C}_m^\lambda$ of $\Gamma(3,2)$ with the property that $\mathcal{W}(\mathcal{C}_m^\lambda) = 3$. Recall that these modules satisfy the property $\tau_{\mathcal{K}_2}(X_m^\lambda) = X_m^\lambda$ and the corresponding Auslander-Reiten sequences in $\mathrm{mod}\,\mathcal{K}_2$ are of the form

$$0 \rightarrow X_m^\lambda \rightarrow X_{m+1}^\lambda \oplus X_{m-1}^\lambda \rightarrow X_m^\lambda \rightarrow 0,$$

where $X_0^\lambda := 0$.
In view of Lemma 5.3, we have a lifted Auslander-Reiten sequences of the form

$$\Sigma: \quad 0 \to \begin{pmatrix} X_m^\lambda \\ \mathrm{Hom}_{\mathcal{K}_2}(M_{3,2}^{(2)}, X_m^\lambda) \end{pmatrix}_{\mathrm{id}} \to E' \to \begin{pmatrix} X_m^\lambda \\ 0 \end{pmatrix}_0 \to 0$$

in $\mathrm{mod}\, B(3,2)$ and dually, we have an Auslander-Reiten sequences

$$\mathrm{D}\,\Sigma : 0 \to \mathrm{D}\begin{pmatrix} X_m^\lambda \\ 0 \end{pmatrix}_0 \to \mathrm{D}\,E' \to \mathrm{D}\begin{pmatrix} X_m^\lambda \\ \mathrm{Hom}(M_{3,2}^{(2)}, X_m^\lambda) \end{pmatrix}_{\mathrm{id}} \to 0$$

in $\mathrm{mod}\, B(3,2)$.

Proposition 5.10. *Let $\lambda \in k \cup \infty$. There is an isomorphism of $B(3,2)$-modules*

$$\mathrm{D}\begin{pmatrix} X_m^\lambda \\ \mathrm{Hom}(M_{3,2}^{(2)}, X_m^\lambda) \end{pmatrix}_{\mathrm{id}} \cong \begin{pmatrix} X_m^\lambda \\ \mathrm{Hom}(M_{3,2}^{(2)}, X_m^\lambda) \end{pmatrix}_{\mathrm{id}}.$$

Proof. Recall that x_1 and x_2 are the generators of $M_{3,2}^{(2)}$ and recall that $B(3,2) = \mathcal{K}_2[M_{3,2}^{(2)}]$ is given by the quiver

$$\begin{tikzcd} 0 \arrow[r, bend left, "x_1"] \arrow[r, bend right, dashed, "x_2"'] & 1 \arrow[r, bend left, "\gamma_1"] \arrow[r, bend right, dashed, "\gamma_2"'] & 2 \end{tikzcd}$$

with relations $\gamma_2.x_1 = \gamma_1.x_2$.
Let furthermore $e_1, \ldots, e_m$ be the standard basis of k^m. For $1 \leq i \leq m$, consider the linear maps

$$\varphi_i : (M_{3,2}^{(2)})_0 \to (X_m^\lambda)_0,\ x_1 \mapsto e_i,\ x_2 \mapsto \lambda e_i + (1 - \delta_{i,m})e_{i+1}.$$

These maps are linearly independent and induce morphisms $\hat{\varphi}_i$ of $\mathcal{K}_2$-modules for $1 \leq i \leq m$, since $\gamma_2.(\varphi_i(x_1)) = \gamma_1.(\varphi_i(x_2))$. Due to the fact that $M_{3,2}^{(2)}$ is indecomposable nonprojective, we have an isomorphism $\mathrm{Hom}_{\mathcal{K}_2}(M_{3,2}^{(2)}, X_m^\lambda) \cong \mathrm{Hom}_{\mathcal{K}_2}(\tau M_{3,2}^{(2)}, \tau X_m^\lambda) \cong \mathrm{Hom}_{\mathcal{K}_2}(P(2), X_m^\lambda)$ and hence the set $\{\hat{\varphi}_i \mid 1 \leq i \leq m\}$ is a basis of the m-dimensional space $\mathrm{Hom}_{\mathcal{K}_2}(M_{3,2}^{(2)}, X_m^\lambda)$.
Now consider the $\mathcal{K}_2[M_{3,2}^{(2)}]$-module $\begin{pmatrix} X_m^\lambda \\ \mathrm{Hom}(M_{3,2}^{(2)}, X_m^\lambda) \end{pmatrix}_{\mathrm{id}}$ with dimension vector (m,m,m). As a representation of

$$\begin{tikzcd} 0 \arrow[r, bend left, "x_1"] \arrow[r, bend right, dashed, "x_2"'] & 1 \arrow[r, bend left, "\gamma_1"] \arrow[r, bend right, dashed, "\gamma_2"'] & 2 \end{tikzcd}$$

in view of the above, the action of x_1 on $\mathrm{Hom}_{\mathcal{K}_2}(M_{3,2}^{(2)}, X_m^\lambda)$ corresponds to the map

$$\psi_1 : \mathrm{Hom}_{\mathcal{K}_2}(M_{3,2}^{(2)}, X_m^\lambda) \to (X_m^\lambda)_0,\ \varphi_i \mapsto e_i,$$

whereas the action of x_2 is given by

$$\psi_2 : \mathrm{Hom}_{\mathcal{K}_2}(M^{(2)}_{3,2}, X^\lambda_m) \to (X^\lambda_m)_0,\ \varphi_i \mapsto \lambda e_i + (1-\delta_{i,m})e_{i+1}.$$

The module $\begin{pmatrix} X^\lambda_m \\ \mathrm{Hom}_{\mathcal{K}_2}(M^{(2)}_{3,2}, X^\lambda_m)\end{pmatrix}_{\mathrm{id}}$ thus corresponds to the representation of $B(3,2)$ given by

$$\begin{tikzcd} k^m \arrow[r, bend left, "\mathbb{I}_m"] \arrow[r, dashed, bend right, "\lambda\mathbb{I}_m + J_m"'] & k^m \arrow[r, bend left, "\mathbb{I}_m"] \arrow[r, dashed, bend right, "\lambda\mathbb{I}_m + J_m"'] & k^m \end{tikzcd}$$

This representation is self-dual, which finishes the proof. □

Proposition 5.11. *Let C^λ_m be the connected component of $\Gamma(3,2)$ containing the module $\iota_{\mathcal{K}_2}(X^\lambda_m)$. Then C^λ_m is of type $\mathbb{Z}A_\infty$ as in Definition 3.21 and $\mathcal{W}(C^\lambda_m) = 3$.*

Proof. Due to Lemma 5.3 and Proposition 5.10, the module $\iota_{\mathcal{K}_2}(X^\lambda_m) = \begin{pmatrix} X^\lambda_m \\ 0 \end{pmatrix}_0$ is contained in a τ-orbit of the form

$$\mathcal{O}^\lambda_m \qquad \cdots \bullet \dashleftarrow \mathrm{D}\begin{pmatrix} X^\lambda_m \\ 0\end{pmatrix}_0 \dashleftarrow \begin{pmatrix} X^\lambda_m \\ \mathrm{Hom}(M^{(2)}_{3,2}, X^\lambda_m)\end{pmatrix}_{\mathrm{id}} \dashleftarrow \begin{pmatrix} X^\lambda_m \\ 0\end{pmatrix}_0 \dashleftarrow \bullet \cdots \tag{14}$$

Let us consider the module $\mathcal{X}^\lambda_m = \mathrm{D}\begin{pmatrix} X^\lambda_m \\ 0\end{pmatrix}_0$. We have

$$\Phi^2_3(\mathcal{X}^\lambda_m) = \mathrm{D}\, X^\lambda_m = X^\lambda_m \in \mathcal{F}(2,2)$$

due to Proposition 4.3, while $\Psi^2_3(\mathcal{X}^\lambda_m) \cong \tilde{S}(0)^{\oplus m} \in \mathrm{EIP}(2,2)$. In view of Proposition 3.33, we have $\tau\mathcal{X}^\lambda_m \in \mathrm{EIP}(3,2)$. Now assume that $\mathrm{Hom}(P(2), \tau\mathcal{X}^\lambda_m) = 0$. In view of the fact that $\tau\mathcal{X}^\lambda_m \in \mathrm{EIP}(3,2)$, this yields

$$\tau\mathcal{X}^\lambda_m \cong \mathrm{D}\,\iota_{K_2}(M_{m,2})$$

for some $m > 2$ due to Proposition 4.3. However, in view of Proposition 5.6, we have

$$\tau^{-1}_{B(3,2)}\,\mathrm{D}\,\iota_{K_2}(M_{m,2}) \cong \mathrm{D}\,\tau_{B(3,2)}\iota_{K_2}(M_{m,2}) \cong \mathrm{D}\,\tau_{B(3,2)}\iota_{K_2}\tau^{-1}_{\mathcal{K}_2}M_{m-2,2} \cong \mathrm{D}\,M_{m-2,3},$$

a contradiction. Hence $\mathrm{Hom}(P(2), \tau\mathcal{X}^\lambda_m) \neq 0$ and an application of Lemma 3.36 yields that $\iota_{\mathcal{K}_2}(X^\lambda_m)$ is quasi-simple in the $\mathbb{Z}A_\infty$- component C^λ_m. Due to Proposition 3.11 (*ii*), the modules $\iota_{\mathcal{K}_2}(X^\lambda_m)$, $\tau_{B(3,2)}\iota_{\mathcal{K}_2}(X^\lambda_m)$ and $\mathcal{X}^\lambda_m$ with dimension vectors $(0,m,m), (m,m,m)$ and $(m,m,0)$, respectively, neither satisfy the equal images nor the equal kernels property.
Since $\tau^k\mathcal{X}^\lambda_m \in \mathrm{EIP}(3,2)$ and, dually, $\tau^{-k}\iota_{\mathcal{K}_2}(X^\lambda_m) \in \mathrm{EKP}(3,2)$ for all $k > 0$, we thus obtain $\mathcal{W}(C^\lambda_m) = 3$. □

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Erklärung

Hiermit versichere ich, dass ich die vorliegende Arbeit abgesehen von der Beratung durch den Betreuer meiner Promotion unter Einhaltung der Regeln guter wissenschaftlicher Praxis der Deutschen Forschungsgemeinschaft selbstständig angefertigt habe und keine anderen als die angegebenen Hilfsmittel verwendet habe.

Einige Hauptergebnisse der Dissertation sind in folgender Publikation enthalten:

Worch, Julia. *Categories of modules for elementary abelian p-groups and generalized Beilinson algebras*, Journal of the London Mathematical Society, 2013, doi: 10.1112/jlms/jdt039.

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(Julia Worch)